Contemporary Physics, 1999, volume 40, number 1, pages 57 – 70

Information theory and blackbody radiation

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The foundations of information theory are briefly reviewed. This background is applied to present a self-contained survey on some recent theoretical results on statistical mechanics and thermodynamics of non-equilibrium radiation-matter systems. Such results generalize the Planck and Stefan–Boltzmann laws, and the Wien displacement law. Present and likely future applications in physics, climatology and industrial processes are discussed in some detail.

1. Introduction

Information theory has been used in statistical physics for at least 40 years [1–4]. It is much simpler mathematically than those approaches to equilibrium statistical mechanics that are presented in most textbooks [5–7]. Information theory is very appealing because of its simplicity and, most importantly, because it can be applied to non-equilibrium systems and is therefore a more general theory. A very clear example is that, as we shall see, it generalizes the Planck spectrum to non-equilibrium systems. In fact, information theory is the cornerstone of many recent findings on non-equilibrium statistical mechanics and thermodynamics (see, e.g. [8–22]). The present paper attempts to introduce briefly the necessary background on information theory, and to provide a survey on some of its most recent developments in the field of non-equilibrium statistical mechanics, especially those related to blackbody radiation [18–21]. Therefore, our aim is not here to present a general survey on the present state of information theory. For a recent review, see [23]. Fundamental ideas and classic applications can be found in seminal treatises [24–30] and reports [31–34] on the subject, whereas the connection to irreversible thermodynamics has been analysed in detail in, e.g. [35].

2. Background: information theory and statistical mechanics

Information-theoretical statistical mechanics is probably a relatively novel formulation to many readers of this journal. In this section we introduce it by providing a brief, intuitive route that is free from any mathematical complexity. The reader who is familiar with information theory can jump to section 3, since the purpose of section 2 is to make the present paper accessible, without the use of any additional material, to scientists working in other fields as well as undergraduate students.

2.1. *Probability and information*

Information theory is a mathematical theory of communication. It was presented by Shannon in 1948 [36]. The key result from Shannon's work is the following. Let us consider a system with n possible states ($k=1,2,\ldots,n$) and denote the probability of the system being in state k as p_k. If we know the values of these probabilities $p_1, p_2, \ldots, p_n$, *and nothing else*, is there some way to measure the amount of information we lack in order to know the precise state of the system? What Shannon showed is that there is an unique measure of this information i, and that (aside from an arbitrary constant factor) it is given by the following quantity

$$i = -\sum_{k=1}^{n} p_k \ln p_k \, . \qquad (1)$$

Shannon derived this result from three very reasonable assumptions, namely: (i) i is a continuous function of the p_k; (ii) if all the p_k are equal ($p_k = 1/n$), then i is a monotonic increasing function of n (so that the more possible states, the less information we have about the actual state); and (iii) if a choice between possible states is broken into two successive choices, the original i is the weighted sum of the two individual values of i. We shall refrain from giving the proof of the well-known result (1) [37]. What we would like to present here is a set of simple examples that will make the meaning of equation (1) absolutely clear. The following

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examples will also be used in order to check an important step in section 3.

Let us consider a system formed by two switches. Each switch can have two positions: position 0 (or 'off') and position 1 (or 'on'). A state of the system is determined if the positions of both switches are given. For example, 01 is the state of the system in which the first switch is off and the second switch is on. The system has therefore four possible states, namely 00, 01, 10 and 11. Let us denote their respective probabilities as p_{00}, p_{01}, p_{10} and p_{11}.

First example: A very simple case is that in which we know the state of the system for sure. For example, assume that we know that the system is in state 00. What amount of information do we lack in order to know the state of the system? In this case we can give the answer without need of any calculation: we do not lack any information. Does equation (1) agree with this? Since in the considered case we have $p_{00}=1$ and $p_{10}=p_{11}=0$, equation (1) yields $i=0$ (note that an indeterminacy appears, so that the l'Hôpital rule may used: $\lim_{p_k\to 0} p_k \ln p_k = \lim_{p_k\to 0} [(\ln p_k)/(1/p_k)] = \lim_{p_k\to 0} [(1/p_k)/(-1/p_k^2)] = 0$. Therefore, equation (1) gives the right result in this trivial case.

Second example: Let us consider a case that is opposite to the previous one. Assume that the probabilities of all states are the same, i.e. $p_{00}=p_{01}=p_{10}=p_{11}=\frac{1}{4}$. This corresponds to the case in which we do not know anything about the state of the system (so that we assign equal probabilities to all possible states). In such a situation, what amount of information would we need in order to determine the state of the system? In this case it is not possible to find out an answer intuitively. All we can say is that this missing information must clearly be higher than before, i.e. $i>0$. However, equation (1) gives a quantitative result, namely $i=-4\frac{1}{4}\ln\frac{1}{4}=\ln 4=1.39$.

Third example: We now consider a case that is intermediate between the two previous ones. Let us assume that $p_{00}=\frac{2}{5}$ and $p_{01}=p_{10}=p_{11}=\frac{1}{5}$. Intuitively, we can only conclude that the missing information must be lower than in the second example (because in that case we did not know anything about the state of the system) and higher than in the first example, i.e. we should have $0<i<1.39$. Equation (1) yields $i=\frac{3}{5}\ln 5+\frac{2}{5}\ln\frac{5}{2}=1.33$, which satisfies that $0<i<1.39$, as it should. These three examples, as well as some generalizations, are visualized in figure 1.

Shannon considered problems in which the values of the p_k are known, and then applied equation (1). Jaynes [2] turned this procedure around by making use of equation (1) in order to determine the values of the probabilities p_k. His criterion is the following: if we have some partial information on the state of the system, the least biased (i.e. most reasonable) probabilities are those such that they maximize the missing information (1): any other probabilities would correspond to assuming additional information, which we do not have. This is the so-called Principle of Maximum Entropy (PME). It is to be noted that the word 'entropy' is not justified at this point, since for the moment we are dealing with a problem of probability theory, not of physics at all. Let us illustrate how the PME works.

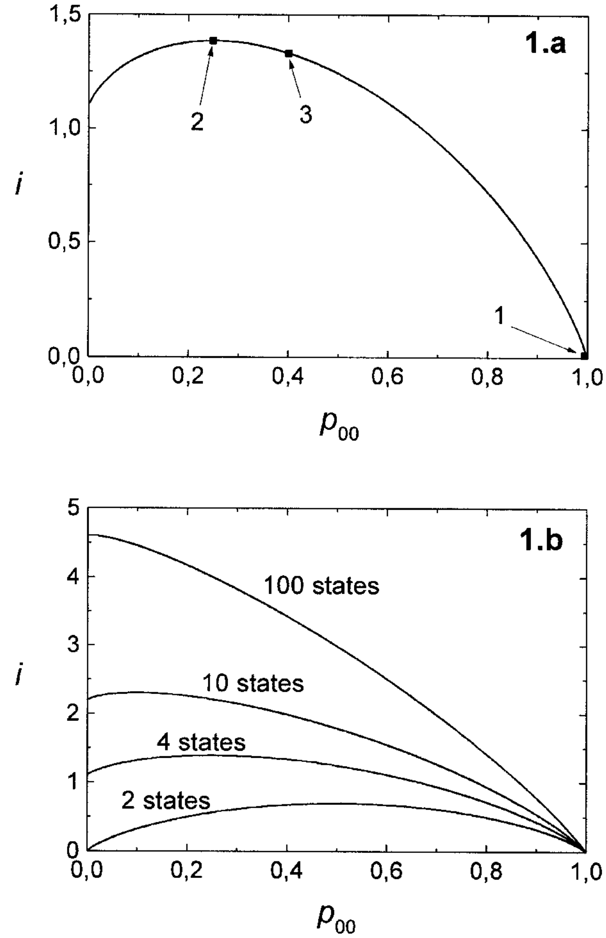

Figure 1. (*a*) is a plot of the missing information *i* for a system with four possible states and $p_{01}=p_{10}=p_{11}$, as a function of the probability p_{00}. Point 1 corresponds to the first example discussed in the text: the state of the system is known, thus there is no missing information. Point 2 corresponds to the second example: in a state of absolute ignorance the lack of information is maximum. Point 3 exemplifies a state of incomplete knowledge. (*b*) generalizes the plot in figure 1 (*a*) to systems with a different number of states. The more possible states, the higher the lack of information is (for a given value of p_{00}). On the other hand, if the number of states *n* is given, the missing information is maximum for $p_{00}=1/n$ (thus all probabilities are the same).

Fourth example: We consider the same system as the three previous examples. Assume, however, that we do not know the values of the probabilities p_k, but we only know that state 00 has probability $p_{00}=\frac{1}{2}$. What are the values of p_{01}, p_{10} and p_{11}? At first glance there is no way to solve this problem: we only know that $p_{00}=\frac{1}{2}$ and, clearly, that $p_{00}+p_{01}+p_{10}+p_{11}=1$. We apparently need two more equations in order to go ahead, i.e. there is some information we lack. However, we can make a reasonable (in fact, the most reasonable) determination of the probabilities under this state of incomplete knowledge: we apply the PME by maximizing the missing information (1) under the conditions $\sum_k p_k - 1=0$ and $p_{00}-\frac{1}{2}=0$. The solution to this variational problem can be easily found out by means of standard calculus techniques [38] by setting $\partial/\partial p_j\,(-\sum_k p_k \ln p_k - \mu\,[\sum_k p_k - 1] - \nu[p_{00}-\frac{1}{2}]) = 0$, with μ and ν being Lagrange multipliers. This yields $p_{00}=\exp(-1-\mu-\nu)$ and $p_j=\exp(-1-\mu)$ (for $j\neq 00$). Since $p_{00}=\frac{1}{2}$, we obtain $\exp -\nu = \exp(1+\mu)/2$, so the normalization condition $\sum_k p_k = 1$ yields $p_j=\frac{1}{6}$ (for $j\neq 00$). This result was to be expected intuitively as the least-biased one: under the specific state of incomplete information considered, there is no reason to assume any difference between the three probabilities p_{01}, p_{10} and p_{11}. Note that if we had considered, instead of the constraint $p_{00}=\frac{1}{2}$, a more complicated one, e.g. $\sum_k k\,p_k = 1.83$, it would not have been possible to solve the problem intuitively, whereas the PME provides a unique, simple method to cope with such cases.

2.2. *Applicability to statistical mechanics*

Up to now we have dealt with probability theory. The value of the PME in statistical mechanics was shown by Jaynes in 1957 [2], who was motivated by the desire to free statistical mechanics from its often apparent dependence on physical hypotheses. Jaynes noted that macroscopic experiments impose constraints on the microstates of physical systems, and that such constraints can be written as expectation values, i.e. as expressions of the form $\sum_k p_k f(k)$, with $f(k)$ the value of a property of state k (e.g. the energy of the system if in state k). Let us illustrate this by considering a matter gas. Let $n(\mathbf{x},t)$ stand for the number density of molecules at position $\mathbf{x}$ and time t. We define the distribution function $f_{\mathrm{m}}(\mathbf{x},\mathbf{p}_m,t)$ by requiring that the quotient $f_{\mathrm{m}}(\mathbf{x},\mathbf{p}_m,t)/n(\mathbf{x},t)$ gives the probability that a particle of the system that has position in d^3x (centred at $\mathbf{x}$ has also momentum in d^3p_m (centred at $\mathbf{p}_m$, divided by d^3p_m/h^3, with h the Planck constant. This probability is normalized, i.e. $\int(\mathrm{d}^3p_m/h^3)(f_m(\mathbf{x},\mathbf{p}_m,t)/n(\mathbf{x},t)) = 1$ (it is not necessary to include h^3, though in this way f_m is dimensionless). The positions and velocities of the molecules of the system are not fixed by the probability distribution $f_m(\mathbf{x},\mathbf{p}_m,t)/n(\mathbf{x},t)$. Indeed, the missing information in order to know the microstate of the system is, from equation (1),

$$i = -\int \frac{\mathrm{d}^3p_m}{h^3}\frac{f_m}{n}\ln\frac{f_m}{n}, \tag{2}$$

where the discontinuous probabilities p_k have been replaced by the continuous probability distribution f_m/n. If an experimentalist can control the number density n and internal energy density u_m of the gas, namely

$$n = \int \frac{\mathrm{d}^3p_m}{h^3} f_m, \tag{3}$$

$$u_m = \int \frac{\mathrm{d}^3p_m}{h^3}\frac{p_m^2}{2m} f_m \equiv \frac{3}{2}nkT \tag{4}$$

(k is the Boltzmann constant and T the temperature [39]), then it is clear from the explanations in the previous section that the least biased distribution function f_m will be the one that maximizes the missing information (2), subject to the constraints (3) and (4). This yields

$$f_m = n\,\exp\left(-1-\tilde{\lambda}n-\tilde{\beta}\,\frac{np_m^2}{2m}\right), \tag{5}$$

with $\tilde{\lambda}$ and $\tilde{\beta}$ Lagrange multipliers. They may be easily found out by substitution of equation (5) into (3) and (4) and integration. This leads to

$$f_m = h^3 n\left(\frac{1}{2\pi m\,kT}\right)^{3/2}\exp\left(-\frac{p_m^2}{2m\,kT}\right), \tag{6}$$

which is nothing but the Maxwell–Boltzmann distribution [6], a well-known result also from other methods that are presented in any textbook on standard statistical mechanics or the kinetic theory of dilute gases.

3. Non-normalized probabilities. Radiation gas

Jaynes [2], as well as other authors (see, e.g. [40] and references therein), remarked that the maximum entropy approach is not restricted to equilibrium situations, and applied it to derive more general results. An interesting possibility which has been considered in the literature [9,41,42] is to generalize the Planck distribution to non-equilibrium states. Let us therefore consider a system that contains radiation. This situation is very different from that in the previous section, because the number density of photons cannot be chosen at will. We can see this as follows. Consider a solid enclosure containing a gas. If the temperature is sufficiently high, the enclosure will shine because of the radiation it emits. We can remove gas molecules until the enclosure is (almost) empty of gas, but not of photons, because the internal walls emit photons into the cavity. Thus the number density of photons is not a quantity that may be independently controlled by the experimentalist. It is true that she can do so if she alters the temperature, but this means changing the internal energy, which has already been taken into account as a constraint on the system (see, e.g. equation (4)). Therefore, in the case of radiation we need to set up a different framework. The one that is presented below is new: it makes explicit use of non-normalized probabilities. Moreover, it can also be applied to matter systems.

Let us go back to the information-theoretical discussion presented in section 2. There it was assumed that the probabilities were normalized, i.e. $\sum_{k=1}^{n} p_k = 1$, simply because this is the most familiar case. However, it turns out that the requirement of normalization is not a necessary condition for the applicability of information theory. After this is shown, we shall see that the description of radiation and matter becomes unified, and that such a unified statistical framework is essential to recent research on non-equilibrium radiative systems.

The question is now: if we define a non-normalized probability P_k as

$$P_k = C\, p_k \tag{7}$$

(with $C > 0$ an arbitrary constant), so that we have $\sum_{k=1}^{n} P_k = C$ instead of $\sum_{k=1}^{n} p_k = 1$, is the equation corresponding to (1), namely

$$I = -\sum_{k=1}^{n} P_k \ln P_k , \tag{8}$$

still a valid measure of the amount of information we lack in order to know the precise state of the system? Substitution of equation (7) into (8) and use of equation (1) yields

$$I = -C \ln C + C\, i , \tag{9}$$

which implies that I is a growing function of i. This means that I measures the amount of information with another scale of measurement. This can be easily checked by going back to the three first examples in the former section: making use of equation (9) (or (8)), we find $i= -C \ln C$ in the first example, $I= -C \ln C+ 1.39\, C$ in the second one, and $I= -C \ln C+ 1.33\, C$ in the third example. Thus the value of I in the third example is higher than in the first one and lower than in the second one, as expected. We therefore see that it is not in fact necessary to deal with normalized probabilities in information theory: making use of non-normalized probabilities leads to different ways to measure the amount of missing information. But if one amount of information is higher than another one, this does not change with the use of non-normalized probabilities. Since, as we have explained, one is interested in the probability distribution that yields the maximum possible value for this missing information, the results will not be affected. Thus we may interpret the single-particle distribution function $f(\mathbf{x},\mathbf{p}_m,t)$ and the quotient $f(\mathbf{x},\mathbf{p}_m,t)/n(\mathbf{x},t)$ as a non-normalized probability and a normalized probability, respectively. We shall make use of these results in order to present a statistical-mechanical procedure that copes both with matter and radiation systems. We stress that in the case of radiation, no *a priori* normalization condition can be imposed.

It is simple to repeat the calculation leading to the Maxwell–Boltzmann distribution (6) making use of an equation analogous to equation (8), namely

$$I = -\int \frac{d^3 p_m}{h^3} f_m \ln f_m , \tag{10}$$

instead of equation (2): maximization of the missing information (10) under the constraints (3) and (4) leads to the Maxwell–Boltzmann distribution (6), as expected. It is also interesting to compare equation (10) with the Boltzmann formula for the entropy density in the case considered (a classical ideal gas), namely [6]

$$s_m = -k \int \frac{d^3 p_m}{h^3} f_m \ln f_m , \tag{11}$$

which shows that $s_m = kI$, i.e. that the entropy density is a measure of that information on the microstate which is not provided by the distribution function.

Let us now generalize the case from that of a purely matter system and tackle an equilibrium problem: consider a blackbody cavity containing an ideal matter gas. Both the walls and the gas inside the cavity will emit and absorb photons. We are interested in deriving the distribution functions both for the matter and the radiation gases inside the cavity. In this case the entropy density is given by the well-known expression [6]

$$s = s_m + s_r = -k \int \frac{d^3 p_m}{h^3} f_m \ln f_m + 2k \int \frac{d^3 p_r}{h^3} [(1+f_r)\ln(1+f_r) - f_r \ln f_r] \tag{12}$$

where the sub-indices m and r stand for matter and radiation, respectively. The energy density of the system is

$$u = u_m + u_r = \int \frac{d^3 p_m}{h^3} \frac{p_m^2}{2m} f_m + 2 \int \frac{d^3 p_r}{h^3} p_r c f_r , \tag{13}$$

where use has been made of the fact that the energy of a photon is $p_r c$, with p_r its momentum and c the speed of light *in vacuo*. The factor 2 in each of the second terms in equations (12) and (13) appears because of the two independent polarizations of the photon [6]. We have seen that the most probable distribution functions are those which maximize the entropy density under the macroscopic constraints that can be controlled by an experimentalist. Thus we must maximize the entropy density (12) under the energy density constraint (13) and the constraint on the matter number density (3) (but not under any additional constraint on the radiation number density because, as stressed above, it cannot be controlled independently by an experimentalist). In this way one easily finds out the distribution functions

$$f_m = \exp\left(-1-\lambda-\beta\frac{p_m^2}{2m}\right), \quad (14)$$

$$f_r = \frac{1}{\exp(\beta p_r c) - 1}, \quad (15)$$

with λ and β Lagrange multipliers. As in the example in the previous section, they can be determined by insertion of the matter distribution function (14) into equations (3) and (4) and integration. In this way, the distribution functions (14) and (15) become

$$f_m = h^3 n\left(\frac{1}{2\pi m kT}\right)^{\frac{3}{2}}\exp\left(-\frac{p_m^2}{2m kT}\right), \quad (16)$$

$$f_r = \frac{1}{\exp(p_r c/kT) - 1}, \quad (17)$$

which are the Maxwell–Boltzmann and Planck distribution functions, respectively, as expected. This confirms that, within information theory, use may be made of non-normalized probabilities. It is on this basis that the recent results mentioned in section 1 may be understood, and the rest of the present paper is devoted to survey them.

4. Non-equilibrium blackbody radiation

If the temperature of a blackbody is uniform, the radiation it emits is Planckian, in agreement with the predictions of equilibrium statistical mechanics. Recent research has been directed towards generalizing this result [18 – 21]: after all, most (if not all) of the radiation we observe has not been emitted by uniform-temperature blackbodies! The first step is to consider a blackbody cavity in a non-equilibrium state: as depicted in figure 2, we assume that the temperature is not uniform. We will here, for the sake of simplicity, neglect heat convection by assuming that the temperature increases upwards ($T_B > T_A$). Heat conduction inside the cavity may also be neglected if the thermal conductivity of the matter gas inside the enclosure is low enough. However, heat will certainly be transported by radiation: since the temperature is higher in the upper part of the enclosure, we expect intuitively that the internal walls will emit more radiation (per unit area and time) in the upper part than in the lower one. Thus we expect a radiative energy flux **F** in the direction shown in figure 2, i.e. **F** = $(0,0,F)$, with $F > 0$ (the z-axis is taken positive downwards). This energy flux **F** is essentially a sum of the photon energies $p_r c$ (per unit volume) times their velocity **c**, i.e. [43]

$$F = 2\int \frac{d^3 p_r}{h^3} p_r c c_z f_r, \quad (18)$$

with c_z the z-component of **c**. The radiative heat flux F is a quantity that can be controlled experimentally, since we

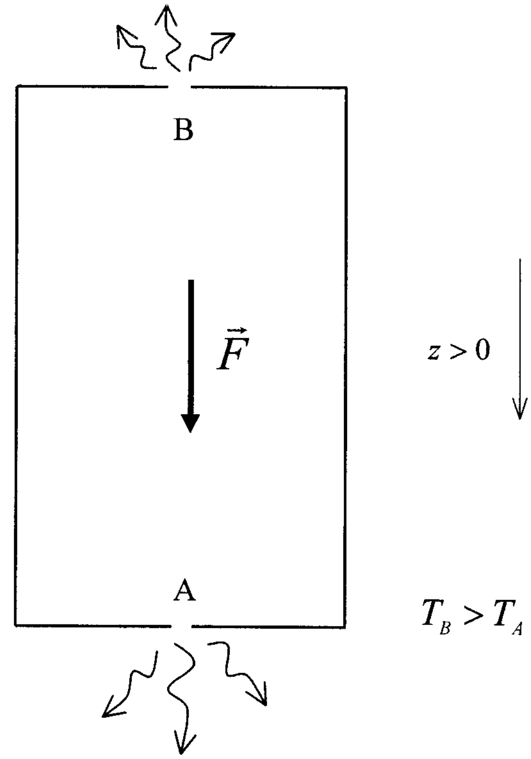

Figure 2. Cavity containing a matter ideal gas. If the temperature is not uniform but increases upwards, there is a radiative heat flux *F* in the direction shown in the figure. The cavity may in general have some small apertures, such as those shown in the figure (A and B), which are crossed by photons leaving and entering the cavity. In case the enclosure does have one or several apertures, the radiation leaving the cavity may be detected by an external observer. Then the internal walls have to be assumed highly absorbing in order to fulfill the definition of a blackbody (i.e. in order to ensure that radiation going into the cavity is not reflected and leaves it again).

have seen that it should be related to the temperature gradient, which may be externally modified by heating mechanisms (e.g. electrical resistances). Thus in order to find out the most probable distribution functions for this system, we can maximize its entropy density (12) under the same constraints as before, namely those given by equations (13) and (3), and the additional constraint (18). This yields for f_m the same result as before (equation (14)), which when used in equations (3) and (4) leads again to the matter distribution (16): as far as f_m is concerned, the only difference with the case dealt with in the former section is that now the distribution (16) is locally Maxwellian, because T depends on position. On the other hand, for the radiation distribution function f_r we obtain

$$f_r = \frac{1}{\exp\left(\frac{p_r c}{kT} - \gamma p_r c c_z\right) - 1} = \frac{1}{\exp\left(\frac{p_r c}{kT} - p_r c\gamma \cdot \mathbf{c}\right) - 1}, \quad (19)$$

with $\gamma = (0,0,\gamma)$ the Lagrange multiplier corresponding to constraint (18). There are several ways to simplify the problem of relating γ to measurable quantities [18,21]. We will present here an extremely simple derivation, because several of the results thus obtained will be useful in the context of applications (to be discussed in the next section). The Planck distribution corresponds to the limit $\gamma \to 0$ in equation (19). Thus states which are sufficiently near equilibrium can be described simply by performing the first-order Maclaurin expansion in γ of the distribution (19),

$$f_r^{(1)} = \frac{1}{\exp(p_r c/kT) - 1}\left(1 + \frac{\exp(p_r c/kT)}{\exp(p_r c/kT) - 1}\gamma p_r c c_z\right). \tag{20}$$

In practice, one is always interested in the radiation intensity, namely [44],

$$I_\nu = \frac{2h\nu^3}{c^2} f_r, \tag{21}$$

ν being the frequency. I_ν is an energy per unit area, time and solid angle, and it can be directly measured by means of a spectrophotometer. If the system is in a steady state, the intensity must satisfy the usual radiative transfer equation [43,45],

$$\frac{c_z}{c}\frac{\mathrm{d}I_\nu}{\mathrm{d}z} = -\sigma I_\nu + j_\nu, \tag{22}$$

where j_ν and σ are the emission and absorption coefficients, respectively (for the sake of mathematical simplicity, scattering has not been included here and σ has been assumed independent of frequency—the latter assumption is the gray approximation [43,45]). Multiplication of equation (22) by c_z/c and integration over all possible values of the photon frequency ν and direction of motion $\mathbf{c}/c \equiv \widehat{\Omega}$ yields, after taking into account that $d^3p_r = p_r^2\,\mathrm{d}\Omega = (h^3\nu^2/c^3)\,\mathrm{d}\nu\,\mathrm{d}\Omega$ (with $\mathrm{d}\Omega$ the differential of solid angle),

$$c\frac{\mathrm{d}P_{rzz}}{\mathrm{d}z} = -\sigma F, \tag{23}$$

with F given by equation (18) and we have defined

$$P_{rzz} \equiv \frac{2}{c}\int \frac{d^3p_r}{h^3} p_r c_z^2 f_r. \tag{24}$$

Use of the distribution function (20) into (24) and (18) and integration yields respectively

$$P_{rzz}\frac{aT^4}{3}, \tag{25}$$

$$F = \frac{4ac^2kT^5}{3}\gamma, \tag{26}$$

with $a = 8\pi^5k^4/15c^3h^3$ the blackbody constant (the Stefan–Boltzmann constant is $ac/4$). Finally, insertion of equations (25) and (26) into (23) allows us to relate γ to measurable quantities as

$$\gamma = -\frac{1}{\sigma ckT^2}\frac{\mathrm{d}T}{\mathrm{d}z}, \tag{27}$$

so that the radiation distribution function (20) reads

$$f_r^{(1)} = \frac{1}{\exp(p_r c/kT) - 1}\left(1 + \frac{\exp(p_r c/kT)}{\exp(p_r c/kT) - 1}\frac{\left|\frac{\mathrm{d}T}{\mathrm{d}z}\right|}{\sigma kT^2} p_r c_z\right), \tag{28}$$

where we have taken into account that in the situation depicted in figure 2 we have assumed that the temperature increases upwards and the z-axis is defined positive downwards, so that $\mathrm{d}T/\mathrm{d}z < 0$. Let us mention that the derivation of equation (28) we have just presented is much simpler than the original one [18].

Before going into some applications, we may note the following properties of equation (28): (i) in equilibrium ($\mathrm{d}T/\mathrm{d}z = 0$) it reduces to the Planck distribution function (17), as expected; (ii) it depends on the photon direction of motion through c_z and is therefore anisotropic, in contrast with the Planck distribution (17): this was to be expected intuitively, since it corresponds to the fact that, in the non-equilibrium situation depicted in figure 2, we have $F > 0$, and this implies that there are more photons moving downwards ($c_z > 0$) than upwards ($c_z < 0$), which is consistent with equation (28); (iii) the distribution (28) is closer to Planckian the higher the absorption coefficient of radiation by matter σ is: this corresponds to the fact that if σ is high, then more photons are absorbed by matter, their energy being re-emitted isotropically (the last term in the radiative transfer equation (22) corresponds to the emission and does not depend on I_ν, i.e. is independent of direction).

In order to illustrate the difference between the Planck distribution (17) and its generalization (28), we make use of equations (21) and (28), and integrate in order to obtain the intensity due to photons that, coming from all possible directions, leave the cavity in figure 2. This yields for the spectral intensity fluxes through apertures A and B in figure 2 [18]

$$i_{\lambda A} = i_{\lambda\,\mathrm{Planck}}(T_A) + i_\lambda^{(1)}(T_A), \tag{29}$$

$$i_{\lambda B} = i_{\lambda\,\mathrm{Planck}}(T_B) - i_\lambda^{(1)}(T_B), \tag{30}$$

where

$$i_{\lambda\,\mathrm{Planck}}(T) = \frac{2\pi c^2 h}{\lambda^5}\frac{1}{\exp(hc/kT\lambda) - 1} \tag{31}$$

is the usual Planckian (or equilibrium) spectral intensity flux, and

$$i_\lambda^{(1)}(T) = \frac{2hc}{3k\sigma T^2\lambda}\frac{\exp(hc/kT\lambda)}{\exp(hc/kT\lambda) - 1}\left|\frac{\mathrm{d}T}{\mathrm{d}z}\right| i_{\lambda\,\mathrm{Planck}}(T)$$

corresponds to the non-equilibrium correction. In the absence of a temperature gradient, the second term in the right-hand side of equations (29) and (30) vanishes and we recover Planck's blackbody spectrum (31). If expansion (20) had been carried out up to second order, we would have obtained an additional term in $|dT/dz|^2$ [18]. This has consequences related to the meaning of temperature in non-equilibrium systems. They will be discussed in section 5.3, but for the moment let us consider situations such that second- and higher-order terms may be neglected. In figure 3 we plot the spectra (29) and (30) for a specific case. It may be tested that higher-order terms are indeed negligible [18], and the maximum corrections with respect to the Planck spectrum (dashed line in figure 3) are of about 8.5%. Such high corrections show that the predictions of information theory are susceptible to being tested experimentally. This example is in contrast with the belief according to which information theory has not led to concrete new results beyond those already obtained by equilibrium approaches. Such a belief was reasonable some time ago [25], but is still widespread at present. On the other hand, looking at figure

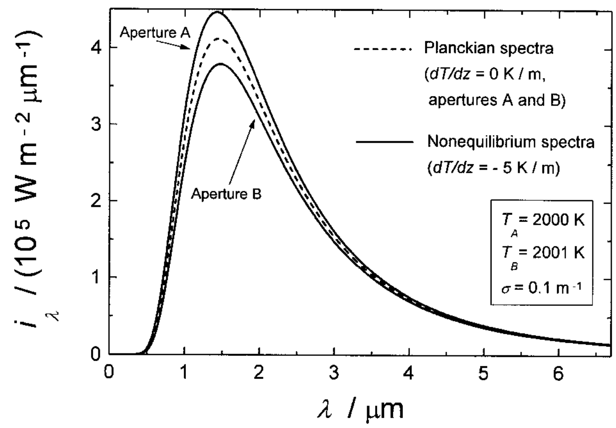

Figure 3. **Spectra of the radiation emitted through the apertures A and B of the blackbody enclosure shown in figure 2. The dashed curve is included for comparison and corresponds to equilibrium (or Planckian) spectra with temperatures T_A and T_B (both spectra are not distinguishable from each other in the figure). Full lines correspond to non-equilibrium spectra (equations (29) and (30)).**

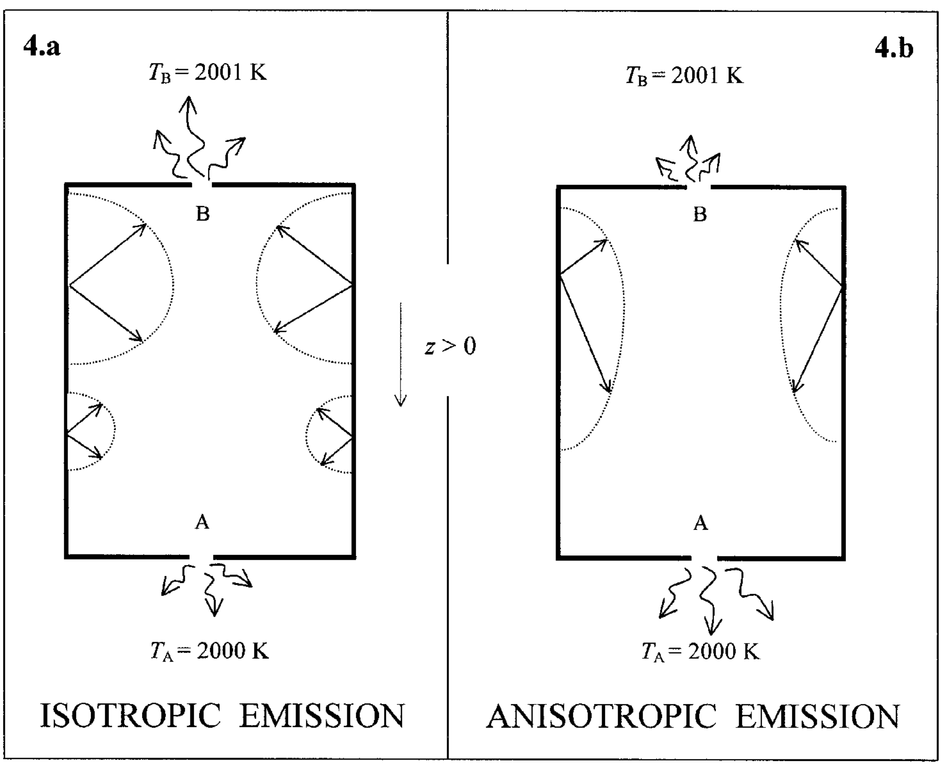

Figure 4. **The figure at the right corresponds to the physical state of the system in figure 2, and illustrates the explanation of figure 3 given in the text. Arrows represent the magnitude of the radiation intensity at a given point and direction. The emission is anisotropic and the radiation leaving the cavity through aperture A is more intense than through B. As explained in the text, it is very important to note that the figure at the left gives a confusing description of radiative emission.**

3 the reader may be surprised that the intensity of radiation leaving the cavity shown in figure 2 may be higher for aperture A than for aperture B: after all, we have assumed that $T_B > T_A$, and we may expect hotter regions to emit a more intense radiation (see figure 2). This reasoning would be right if the emission were isotropic (see figure 4 (*a*)). However, according to equation (28) there are more photons moving downwards ($c_z > 0$) than upwards ($c_z < 0$). Thus the radiation intensity is not only inhomogeneous but also anisotropic (see figure 4 (*b*)), and this explains the fact that more photons cross aperture A than aperture B per unit time. A simple illustration is provided by a small part of a star: in such a physical system, the downward direction in figures 2 and 4 (decreasing-temperature direction) corresponds to the outward radial direction, which is the direction of maximum radiation intensity, in agreement with figure 4 (*b*).

Integration of equations (29) and (30) over all possible wavelengths yields the following results for the total energy fluxes $q = \int_0^\infty i_\lambda \, d\lambda$ leaving the cavity in figure 2 [18]

$$q_A = \frac{ac}{4} T_A^4 + \frac{2ac}{3\sigma} T_A^3 \left| \frac{dT}{dz} \right|,$$

$$q_B = \frac{ac}{4} T_B^4 - \frac{2ac}{3\sigma} T_B^3 \left| \frac{dT}{dz} \right|,$$

where $a = \pi^2 k^4 / 15 c^3 \hbar^3$ is the blackbody constant ($\hbar = h/2\pi$ is the reduced Planck constant). In equilibrium ($dT/dz = 0$), the former expressions reduce to the Stefan–Boltzmann law ($q = ac/4T^4$), as expected. These results are illustrated in figure 5, where we observe that the higher the value of the temperature gradient is, the more important

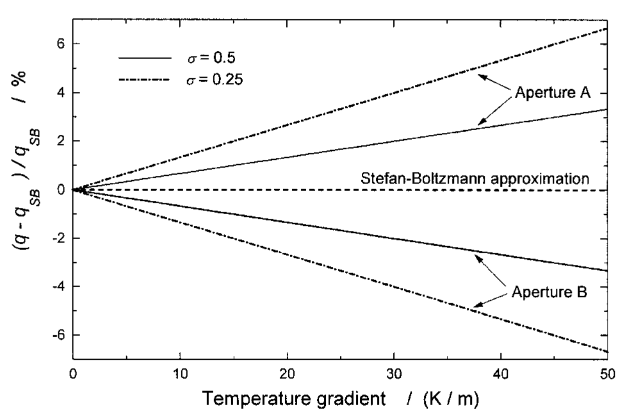

Figure 5. Total heat flux (i.e. integrated intensity of radiation) leaving the cavity in figure 2, as a function of the temperature gradient and for two values of the absorption coefficient σ. The temperature at the considered aperture (A or B) is 2000 K in all cases depicted. The plotted quantities are fluxes relative to those which follow from the Stefan–Boltzmann law ($q = (ac/4)T^4$), which only in the case of equilibrium ($dT/dz = 0$) is an exact

non-equilibrium corrections become. From figure 5 we can also note that a lower value for the absorption coefficient σ corresponds to further away from equilibrium state, in agreement with the discussion given below equation (28).

5. Applications

5.1. *The Wien displacement*

An important statistical-mechanical result for equilibrium radiation is Wien's displacement law, which follows directly from Planck's spectrum (7) and states that the wavelength λ_{max} of maximum blackbody radiation is given by (see, e.g. [6])

$$\frac{ch}{kT\lambda_{max}} = 4.9651 . \quad (32)$$

This equilibrium law can be generalized to non-equilibrium states by making use of the spectral distributions (29) and (30) instead of the Planck distribution (31). We quote the results for the apertures A and B shown in figure 2, which are [19]

$$\frac{ch}{kT_A \lambda_{max\,A}} = 4.9651 + 3.3101 \frac{\left|\frac{dT}{dz}\right|}{\sigma T_A}, \quad (33)$$

$$\frac{ch}{kT_B \lambda_{max\,B}} = 4.9651 - 3.3101 \frac{\left|\frac{dT}{dz}\right|}{\sigma T_B}, \quad (34)$$

with T_A and T_B the temperatures at the apertures A and B in figure 2, respectively. Equation (33) predicts that outside equilibrium, an observer who detects the radiation looking along the direction of the temperature gradient (e.g. an observer located at the aperture A in figure 2) will find a decrease of the wavelength of maximum radiation intensity with respect to the equilibrium value, which is given by Wien's displacement law (32). According to equation (34), an observer looking in the opposite direction (e.g. an observer located at B) will detect an increase for this wavelength. In figure 6 we illustrate the non-equilibrium spectrum through aperture A for several temperatures. In the inset of figure 6 we see that the wavelength of maximum intensity is lower than in the case of an equilibrium system at the same temperature, in agreement with equation (33).

The Wien displacement and the Planck spectrum are very interesting from a conceptual perspective. Moreover, they also have important applications in systems where the temperature is so high that it cannot be measured by means of a contact thermometer. Some examples are furnaces (such as those used in, e.g. high-temperature processing metal industries [46,47]), sonoluminescence [48] and stellar atmospheres [49]. However, in such systems the temperature is not uniform and any description based on equilibrium approximations may be very crude. In order to illustrate this point, let us assume that a small part of

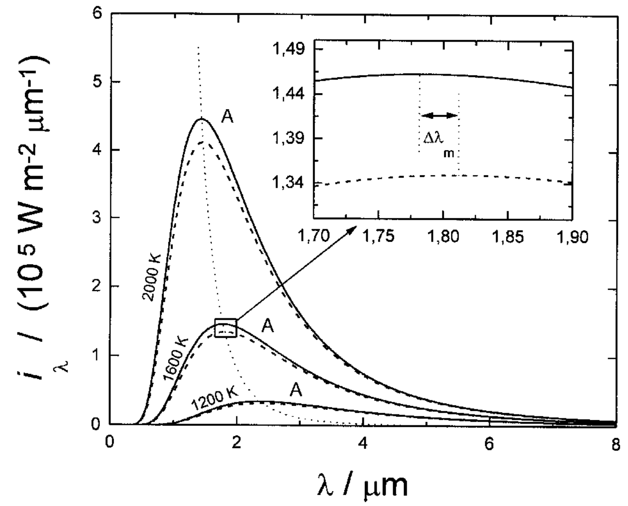

Figure 6. Non-equilibrium (solid curves) spectra for different temperatures and aperture A in figure 2. In all cases $(1/\sigma T_A)(dT/dz) = 0.025$. Planckian spectra are included for comparison (dashed curves). The inset shows the range around the maxima in detail, for a specific temperature. Note the shift $\Delta\lambda_m$ in the maximum, which corresponds to the generalized Wien displacement (33).

some physical system (such as an industrial furnace, a sonoluminescent bubble or an outer layer of a star [19]) can be approximately represented by figure 2. Consider the specific case in which we observe the system looking in the direction of the temperature gradient (aperture A in figure 2). If the values of the relevant parameters are, e.g. $\sigma = 0.1\ m^{-1}$ [18] and $|dT/dz| = 10\ km^{-1}$, and the observed wavelength of maximum intensity is $\lambda_{\max A} = 1.4021\,\mu m$, then equation (33) can be used to estimate a temperature of $T_A = 2000$ K. What would the difference be had we simply applied the Wien displacement law? The estimated temperature would then be obtained from equation (32). This yields 2067 K. Therefore, the difference is of about 3.5%. Such an error should not be neglected, and provides a simple illustration of the relevance of extending the equilibrium description based on the Planck spectral and the Wien's displacement laws.

5.2. *Temperature measurements in shock waves and industrial processes*

It would be rather incomplete to refer here only to Wien's displacement law in the context of radiation-based temperature measurement. Additional, very interesting methods exist. One such method is the so-called temperature colour measurement, which has become very important in the context of shock waves. Shock waves are used in order to obtain experimental information on the equation of state of hot dense matter [50], which is useful, e.g. in inertial confinement fusion research. In some of these experiments, a target made of aluminium is irradiated by a laser. The incident laser radiation causes a shock wave that propagates through the target and gradually heats it up: if the shock wave is strong enough, the target becomes completely vaporized [51]. The rear-face of the target emits a sudden flash at the arrival of the high-temperature wave. This radiation can be recorded and analysed in order to estimate the temperature. Colour temperature estimations rely on the use of filters, which are used in order to measure the radiation intensity at different wavelengths. Consider a case in which a red ($\lambda \approx 600$ nm) and a blue ($\lambda \approx 400$ nm) filter are used [51]. Let us call the corresponding intensities i_r and i_b. The experimental ratio i_r/i_b and Planck's law (31) may be used in order to obtain a rough estimate of the temperature. For example, for $i_r/i_b = 0.49$ this yields a temperature of 0.823 eV. The question is to what extent this result, stemming from an equilibrium hypothesis, is reliable. In fact very high temperature gradients are well-known to appear in shock waves [51,52]. Since in these experiments the radiation that is recorded comes from a cooler plasma that obscures the higher-temperature material behind it [51], the observation takes place in the direction of the temperature gradient and we may therefore carry out a non-equilibrium estimation making use of equation (29). The absorption coefficient can be calculated, and even measured, independently of the shock-wave experiments, and is known to be frequency-dependent [53]. Equation (29) can be applied to this case, with the only change that σ now depends on frequency. We denote the absorption coefficients in the red and blue wavelengths as σ_r and σ_b, respectively. For the same value of i_r/i_b as above, a temperature gradient of $|dT/dz| = 1.3 \times 10^6$ eV m^{-1}, $\sigma_r = 8.9 \times 10^4\ cm^{-1}$ and $\sigma_b = 3.8 \times 10^4\ cm^{-1}$, equation (29) yields a temperature of 0.800 eV. The difference between this result and the estimation based on neglecting the influence of the temperature gradient is therefore of about 3%, which again indicates the usefulness of the non-equilibrium results in a specific application.

Similar estimations can be performed for other optical methods of temperature measurement, such as the ratio thermometers used in some steel and aluminium industries (these thermometers in fact carry out colour temperature measurements, on the same principle as that described above [46], which is also useful in astrophysics [54]), the brightness and emissivity method (which has been used in shock waves [55]), spectral-band thermometry (which is the method on which many radiation thermometers are based, including some fibre-optic thermometers [46]) and total radiation thermometry [46]. The latter three methods are based on direct comparisons to Planckian spectra, but in the case of systems with a non-uniform temperature distribution the emission spectra (such as those plotted in figures 3 and 6) are not Planckian, and from the estimations presented below equation (31) it is clear that non-negligible corrections can easily arise [18].

5.3. *What does a thermometer measure outside equilibrium?*

In sections 5.1 and 5.2 we have discussed the influence of non-equilibrium corrections on radiation temperature measurements. However, up to this point only first-order corrections have been taken into account. We have already mentioned that in general second-order terms are not negligible, so that an additional term appears in equations (29) and (30),

$$i_{\lambda A} = i_{\lambda\,\mathrm{Planck}}(T_A) + i_\lambda^{(1)}(T_A) + i_\lambda^{(2)}(T_A)\,, \qquad (35)$$

$$i_{\lambda B} = i_{\lambda\,\mathrm{Planck}}(T_B) - i_\lambda^{(1)}(T_B) + i_\lambda^{(2)}(T_B)\,, \qquad (36)$$

where $i_\lambda^{(2)}(T)$ corresponds to the second-order correction. This correction has been recently shown to depend on how the temperature is defined [20]. In non-equilibrium systems, there are several ways to define a temperature-like variable. For radiative systems, one way is to assume that the usual equilibrium law for the radiation energy density u_r holds, i.e. [6]

$$u_r = aT_r^4\,. \qquad (37)$$

The parameter T_r is well known to correspond to the temperature, provided that the system is in equilibrium. Is this so also in non-equilibrium states? A general thermodynamic definition for the temperature T is [35]

$$T^{-1} = \frac{\partial s}{\partial u}, \qquad (38)$$

where s and u are the entropy and the energy of the system per unit mass. From the definitions (37) and (38) it follows that different results are obtained if we write, e.g. the intensity (35) in terms of the thermodynamic temperature T_A than if we write it in terms of the local-equilibrium temperature T_{rA} [20]. This is illustrated in figure 7. The meaning of this figure is the following. If use is made of a contact thermometer (in order to determine the temperature) and of a spectrophotometer (in order to measure the radiation intensities) in the system depicted in figure 2, then comparison with the information-theoretical predictions in figure 7 may make it possible to determine experimentally whether T or T_r is the quantity measured by a thermometer outside equilibrium. This is one of the most controversial debates in non-equilibrium thermodynamics at present [56,57]. In this context, the importance of information theory is that it can be applied in order to find out how this subtle question might be solved experimentally [14,20].

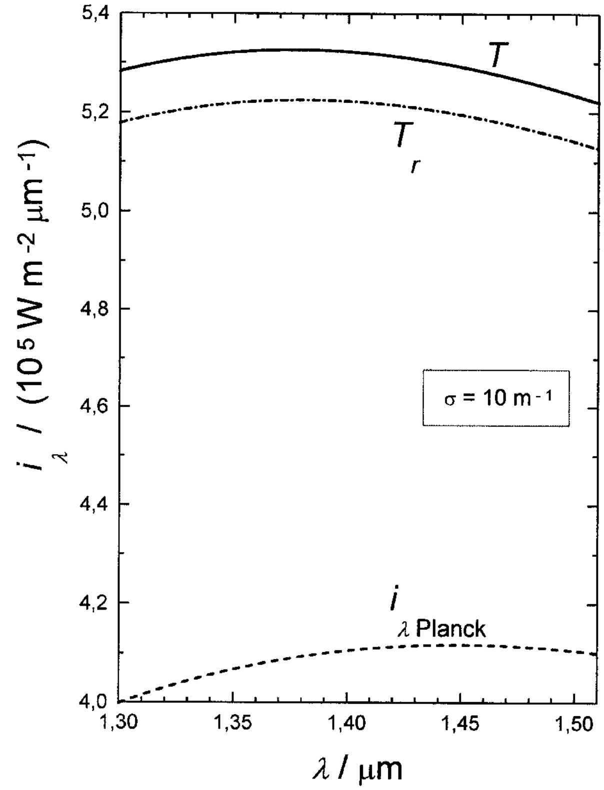

Figure 7. Spectra of the light emitted through aperture A in figure 2 according to the predictions of information theory and assuming T (full line) or T_r (dashed-dotted line) is the measurable temperature. Only a narrow wavelength range near the maxima is shown. The temperature is 2000 K and the temperature gradient is 15 K m^{-1}.

5.4. *Climatology*

From equations (26) and (27) we find that the radiative heat flux is related to the temperature gradient in the following way

$$F = -\lambda \frac{\mathrm{d}T}{\mathrm{d}z}, \qquad (39)$$

with $\lambda = 4acT^3/3\sigma$. This is a Fourier-type equation for heat radiation (instead of the more familiar case of heat conduction). It follows, as we have seen, from information theory, but it is in agreement with some phenomenological descriptions [58]. However, the starting point is always equation (22), which is the radiative transfer equation in the special case of steady states. The radiative transfer equation has also been widely used for systems that evolve in time, and is given by [45]

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \frac{c_z}{c}\frac{\mathrm{d}I_\nu}{\mathrm{d}z} = -\sigma I_\nu + j_\nu\,, \qquad (40)$$

where the first term includes a time derivative and is therefore absent in the case of steady states (equation (22)).

It is not difficult to make use of the information-theoretical result (20) into equation (40), instead of (22). This yields [21]

$$\tau \frac{\mathrm{d}F}{\mathrm{d}t} + F = -\lambda \frac{\mathrm{d}T}{\mathrm{d}z}, \tag{41}$$

where τ has units of time. This result generalizes equation (39) to non-steady states. It has also been derived phenomenologically for extremely anisotropic radiation fields [59,60].

Thermodynamic climate models make it possible to reduce the number of variables and, in this way, to identify the basic parameters affecting the evolution of climate [61]. Because the climate is driven by the radiation field, it has been stressed that the role of radiative heat transport should not be neglected [62]. With this perspective, Pujol and Llebot have analysed the role of the first term in equation (41) (which does not appear in the steady-state equation (39)) on the climate system. In this way, they have been able to analyse the influence of the ratio τ/λ on the response of climatic states with respect to small perturbations [63], thereby enlarging the range of applicability of previous approaches [64]. In this context, it is worth mentioning that use of a generalized equation of the form (41) for the matter heat flux shows that certain values of the relevant parameters can lead to self-sustained periodic oscillations in one-dimensional models of the climate system [65].

Another very interesting approach to the modelling of the climate is based on the rate of entropy production, σ^s. The meaning of σ^s can be understood by saying that it corresponds, in the case of an isolated system, to the entropy increase per unit time and volume. For the specific case in which equation (41) holds, the rate of entropy production is of the form [21]

$$\sigma^s = \frac{F^2}{\lambda T^2}, \tag{42}$$

which satisfies that $\sigma^s \geqslant 0$. This is as expected, because according to the second law of thermodynamics the entropy of an isolated system is an increasing function of time. The application of approaches of this kind to the study of climate goes back to Paltridge, who derived distributions of temperature and cloud-cover from the maximization of the rate of entropy production [61]. Although a rigorous proof of the validity of such a procedure is still lacking, the distributions thus obtained showed a remarkably good agreement with observations. This has motivated many studies on the climate behaviour in which the rate of entropy production is a fundamental point. However, no expression of the type of (42) was used in the original studies. In fact, Paltridge considered only the entropy production corresponding to the matter part of the system, whereas equation (42) is an expression for the radiative part. The role of the radiation entropy production has been recently taken into account, and this has lead to reasonable, more general results for temperature and cloud distributions [66]. More recent models extend this approach in order to predict the role of greenhouse gases, and additional effects, in future climatic scenarios [67].

5.5. *Astrophysics*

Since the photon energy is $p_r c = h\nu$, according to equations (21) and (28) the intensity of radiation has the form

$$I_\nu = a_\nu + b_\nu \cos\theta, \tag{43}$$

were a_ν and b_ν are functions of frequency and $c_z = c\cos\theta$, i.e. θ is the angle between the photon velocity **c** and γ (or between **c** and the temperature gradient, see equation (27)). However, it is to be stressed that an intensity of this form can be expected to give a realistic description of radiation only near equilibrium. Otherwise, use should be made of the distribution (19), which combined with equation (21) yields an intensity of the form

$$I_\nu = \frac{2h\nu^3/c^2}{\exp(A_\nu - B_\nu\cos\theta) - 1}, \tag{44}$$

instead of (43).

It is very interesting that the form (43) for the radiation intensity is well known in astrophysics. It is known as the Eddington approximation, and can be derived by means of phenomenological theories [45,68]. The Eddington approximation (43) corresponds to equations (29) and (30) (for the case of the system depicted in figure 2 instead of the Sun) and is said to give a rather good description of non-

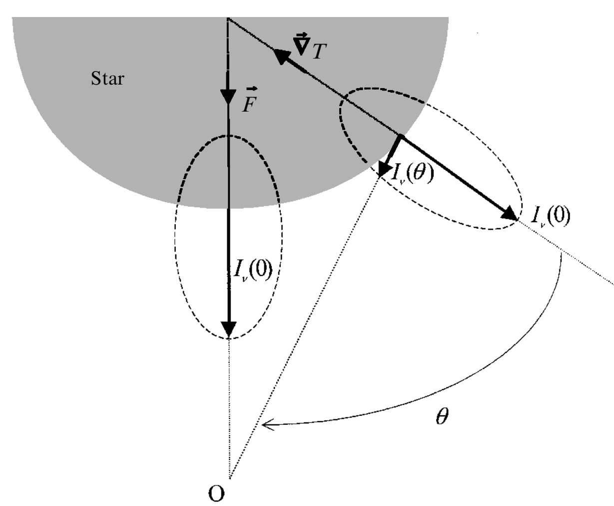

Figure 8. Stellar limb darkening. An observer at O sees less light the higher the angle of sight θ is. This is a consequence of the anisotropy of non-equilibrium radiation. Experimental data are compared to the predictions of two theoretical approaches in figure 9.

equilibrium radiation [45]. However, information theory tells us that this should be so only near equilibrium. We can see this in a very well-known astrophysical application of radiative transfer: the solar limb darkening. This phenomenon is illustrated in figure 8, and it explains why in the photographs of the Sun, the edge appears darker than the centre. According to the Eddington approximation (43), the relative effect should be of the form

$$\frac{I_\nu(\theta)}{I_\nu(0)} = \frac{a_\nu + b_\nu \cos\theta}{a_\nu + b_\nu}, \tag{45}$$

where we have used the current notation in astrophysics and written $I_\nu(\theta)$ instead of I_ν, and $I_\nu(0) = I_\nu(\theta = 0)$ for the maximum intensity. According to equation (45), the Eddington model predicts that a plot of $I_\nu(\theta)/I_\nu(0)$ versus $\cos\theta$ should yield a straight line (for a given frequency). From the experimental data [69] reproduced in figure 9, it is seen that this is a reasonable approximation (the dotted line in figure 9 is a fit to equation (45)). On the other hand, the information-theoretical intensity (44) predicts for the limb darkening

$$\frac{I_\nu(\theta)}{I_\nu(0)} = \frac{\exp(A_\nu - B_\nu) - 1}{\exp(A_\nu - B_\nu \cos\theta) - 1}, \tag{46}$$

which is a more general description than the near-equilibrium approximation (45). The full line in figure 9 is a fit to equation (46). Note that both fits make use of the same number of free parameters (namely, two of them), but the information-theoretical prediction is more accurate. Such a result is very encouraging. In contrast to some well-known approaches [70], it is not based on the phenomenological assumption of radiative local thermodynamic equilibrium (i.e. the approximation $j_\nu \simeq \sigma I_{\nu\,\mathrm{Planck}}$ [45]). This gives a hint about possible future applications of information theory in astrophysics. Here some brief comments may be appropriate. Let us note that B_ν will be related to the temperature distribution in the star (see equation (27), which holds in the simplest model), so that comparisons between information-theoretical predictions, such as (46), and observed limb darkenings can be used, if reliable values for the absorption coefficient are available [71], in order to infer temperature distributions. In fact, more general laws than (46) can be derived by the use of additional constraints besides (18), in complete analogy to similar analysis for matter systems [72]. We have used the Sun as an illustration, but comparison between observed and calculated intensities are of special interest when considering stars for which few observational data are available [73].

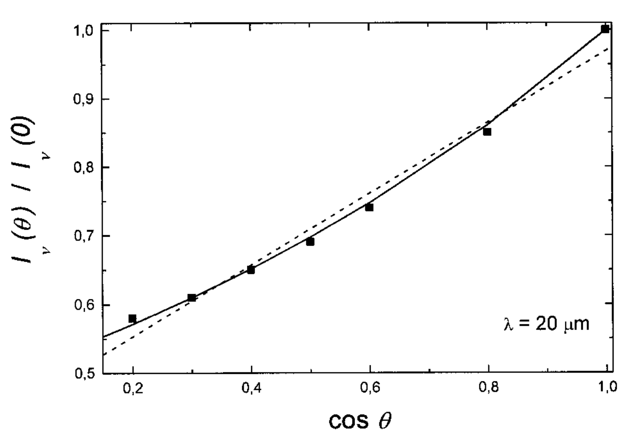

Figure 9. The squares are measurements of the solar limb darkening. The dotted line is a fit of these experimental points to the first-order approximation (equation (45)) and the full line is a fit to the more general information-theoretical result (46).

5.6. *Further applications*

There are many other systems in which non-equilibrium radiation is of importance, although the difficulty of carrying out direct, detailed comparisons between theory and experiments might in some cases make some theoretical predictions of little practical use at present.

One example is sonoluminescence, which consists of very fast light flashes emitted from a liquid under high-intensity sound waves. In this situation, acoustical energy is transformed into radiative energy, temperature gradients are very high and thermal spectra have been reasonably invoked as one possible explanation [48]. In spite of this, the absorption of light by water for wavelengths below 190 nm does not make it possible to map out the complete spectrum [74], and this is an important problem in order to carry out conclusive comparisons between experimental spectra and theoretical ones (such as those predicted by information theory, for some examples see figures 3 and 6).

Before closing this section we would like to mention another example: the cosmic microwave background radiation. It has been proposed that it may be used as a test of generalized Planckian spectra [75], specifically of those that arise in the context of a very interesting generalization of the usual statistical mechanics (in the latter the additivity of the entropy of independent systems holds) [76], but again a practical problem arises, since the measured spectrum is almost Planckian [77] and, in fact, the experimental deviations may be due entirely to instrumental effects [75].

6. Concluding remarks

We have surveyed how Jaynes' information theory leads to non-equilibrium extensions of the well-known theoretical results on radiation due to Planck, Wien and Boltzmann. Some applications have been presented. They can be seen as an expression of our motivation when undertaking this work: most of the radiation around us (in the climate system [62], in the stars [43], in practical blackbodies [46], etc.) is not in equilibrium.

It has also been our intention to persuade the reader of the great predictive potential of information theory. Information theory is simpler than the more traditional approaches, and it can deal with non-equilibrium systems. We would like to emphasize that the results surveyed cannot be derived making use of any statistical-mechanical method of those which appear in textbooks of equilibrium statistical mechanics [5–7], simply because the validity of such methods is restricted to equilibrium situations. In this context, it is interesting to mention that the textbook by Huang [5] includes some very interesting discussions on the method of the most probable distribution, and follows a classical approach (the maximization of the volume in phase space). As noted by Jaynes [78], this is a method advanced by Boltzmann himself, but in order to deal with non-equilibrium systems it should be replaced by the maximization of the entropy density. Jaynes' approach has been summarized in section 2 and shows that, in contrast to what is stated in [5], the method of maximum probability can indeed be applied to obtain predictive results for non-equilibrium systems. An example of such an application is the generalization of the Planck distribution function, as illustrated in figures 3 and 6.

Acknowledgements

The authors would like to thank D. Jou and T. Pujol for discussions. This work has been partially funded by the DGICYT of the Spanish government under grant No. PB96-0451.

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