Eur. J. Phys. **21** (2000) 95–100. Printed in the UK PII: S0143-0807(00)06013-X

How long does it take to boil an egg? A simple approach to the energy transfer equation

P Roura, J Fort and J Saurina

GRM, Departament de Física, Universitat de Girona, E17071-Girona, Catalonia, Spain

Received 14 July 1999, in final form 16 September 1999

Abstract. The heating of simple geometric objects immersed in an isothermal bath is analysed qualitatively through Fourier's law. The approximate temperature evolution is compared with the exact solution obtained by solving the transport differential equation, the discrepancies being smaller than 20%. Our method succeeds in giving the solution as a function of the Fourier modulus so that the scale laws hold. It is shown that the time needed to homogenize temperature variations that extend over mean distances x_{m} is approximately x_{m}^2/α, where α is the thermal diffusivity. This general relationship also applies to atomic diffusion. Within the approach presented there is no need to write down any differential equation. As an example, the analysis is applied to the process of boiling an egg.

1. Introduction

The title of this paper proposes a problem that *a priori* appears quite difficult to solve for two main reasons. First, one should be able to solve a non-elementary three-dimensional problem of non-steady heat conduction. Second, the thermal conductivity and specific heat of the egg are parameters whose values are not currently available from non-specialized sources. In what follows we propose reasonable approximations that allow us to reach a satisfactory result. Along the way, we will obtain an important relationship that applies to non-steady problems involving both heat and mass transport.

In introductory texts in physics, the problem of heat transport (mass diffusion) is addressed by giving Fourier's (Fick's) law in one dimension:

$$J_q = -\kappa \frac{\partial T}{\partial x} \quad \text{(Fourier)} \qquad J_m = -D \frac{\partial c}{\partial x} \quad \text{(Fick)} \tag{1}$$

which states that the heat flux J_q (mass flux J_m) is proportional to the temperature gradient, $\partial T/\partial x$ (concentration gradient $\partial c/\partial x$). The proportionality constant is the thermal conductivity, κ (diffusion coefficient D). These laws are applied to steady-state situations, in which they allow one to calculate the flux once the temperature gradient (or the concentration gradient) is known and *vice versa*. On the other hand, in non-steady situations the temperature (or concentration) changes with time and solutions are much more difficult to find. For instance, the net heat that enters the egg per unit time in our problem will be given by Fourier's law. However, this flux will increase the egg's temperature, diminish the temperature gradient and, consequently, the heat flux. So, we are faced with problems where the temperature will depend both on time and position, involving complex differential equations. In spite of this

0143-0807/00/010095+06$30.00

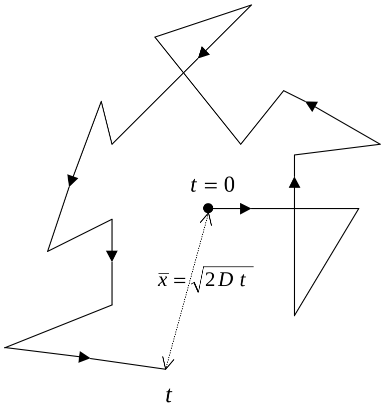

Figure 1. Scheme of the random walk of a molecule due to thermal agitation.

difficulty, something is said concerning non-steady diffusion in introductory texts [1]. As a rough approximation, the time needed to homogenize the concentration variations that extend over mean distances x_m is

$$t_\mathrm{d} \approx \frac{x_\mathrm{m}^2}{D}. \tag{2}$$

This formula is usually justified by the fact that a molecule undergoing a random walk due to thermal agitation will depart from the initial position after a delay t by a mean distance (see figure 1)

$$\overline{x} = \sqrt{2Dt}. \tag{3}$$

Of course, equation (2) is a very nice and general relationship, but its statement needs to be justified by an extra piece of information introduced *ad hoc*, which is very far from the macroscopic phenomenological equations (1) (see [2], which gives a nice view of diffusion at the atomic level). The case is worse in non-steady thermal conduction. This is presumably because, in contrast to the diffusion problem, conduction of heat does not occur by a simple microscopic mechanism similar to the random walk of molecules. However, the phenomenological equations are formally the same and, consequently, one intuitively expects that a relationship equivalent to equation (2) can be stated. Indeed, the analysis of the non-steady problem developed in the next section will lead us to the thermal equivalent of equation (2).

2. Heating bodies of simple geometry

Once the egg is immersed in a pan of boiling water, its surface will acquire, almost instantaneously, the water temperature. This temperature will be homogeneous provided that convection in the water bath due to boiling is sufficiently intense. So, we are faced with the heating of an object with initial internal homogeneous temperature T_0 and surface temperature T_s imposed by the water bath. For simple geometric shapes, such as an infinite plate, an infinite cylinder or a sphere, the exact solution can be obtained by solving the corresponding differential equation.

2.1. Exact solution

The law of conservation of internal energy states that the internal energy density, u, changes according to the divergence of the heat flux $\boldsymbol{J}_q$ [3]:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\boldsymbol{\nabla} \cdot \boldsymbol{J}_q. \tag{4}$$

To solve a particular transport problem, equations are needed for the internal energy density and the heat flux. The variation of the internal energy is related to that of the temperature through

$$du = \rho c_e \, dT \tag{5}$$

where ρ is the mass density and c_e the specific heat. On the other hand, if heat transport is due to conduction, Fourier's law holds (see equation (1)). In vector notation

$$\boldsymbol{J}_q = -\kappa \boldsymbol{\nabla} T. \tag{6}$$

Now, substitution of equations (5) and (6) in equation (4) yields the evolution equation for T:

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T. \tag{7}$$

This equation relates the rate of change of temperature with respect to time at any space point to the temperature distribution and the thermal diffusivity, namely

$$\alpha \equiv \frac{\kappa}{c_e \rho} \tag{8}$$

which collects the parameters necessary for understanding the thermal behaviour of the material. Once the initial temperature distribution and the boundary conditions are known, equation (7) can be solved.

This rigorous approach can be found in specialist texts on heat transport [4]. For example, in the heating of a homogeneous infinite plate of half-thickness x, the temperature at its centre T_d evolves [4] according to

$$\frac{T_\mathrm{s} - T_\mathrm{d}}{T_\mathrm{s} - T_0} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right) \exp\left[-(n\pi/2)^2 F_0\right] \qquad n = 1, 3, 5, \ldots \tag{9}$$

where $T_0 = T_\mathrm{d}(t = 0)$ is the initial temperature and T_s is the temperature of the thermal bath. The solution (9) depends on an important dimensionless parameter

$$F_0 \equiv \frac{\alpha t}{x^2} \tag{10}$$

known as the 'Fourier modulus'. In the case of a cylinder [5] and a sphere [6] instead of a plate, the exact analytical solutions are also given by infinite series (involving Bessel functions for a cylinder [5]). Because of this complexity, they are usually plotted or tabulated. In figure 2 we have plotted the exact temperature evolution at the centre of these objects (for example, the full line labelled 'infinite plate' in figure 2 is a plot of equation (9)). An important conclusion can be drawn from figure 2, namely that the objects are almost in thermal equilibrium with the surrounding bath when $F_0 \approx 1$. This fact and the functional dependence on the Fourier modulus, although straightforward in view of the exact solution, are not easy to understand by looking at the transport equation (7). So, we propose to solve the problem with a simplified method that, besides giving a reasonable quantitative solution, helps one to understand the main points of the problem. In contrast to the conventional method, we do not solve any differential equation. This makes our approach tractable in an introductory course in physics.

2.2. Approximate solution

The thermal evolution of an infinite flat plate will be similar to that shown in figure 3. We want to estimate the time t_d needed to reach a temperature T_d at the centre of the plate. Since we are not interested in the exact spatial distribution of temperature, the problem will be analysed from an integral point of view, rather than the differential one presented in section 2.1. The law of conservation of internal energy (4) between the initial ($t = 0$) and final ($t = t_\mathrm{d}$) states can be written as

$$\frac{\Delta U}{t_\mathrm{d}} = 2A\overline{J}_q \qquad \text{(plate surface).} \tag{11}$$

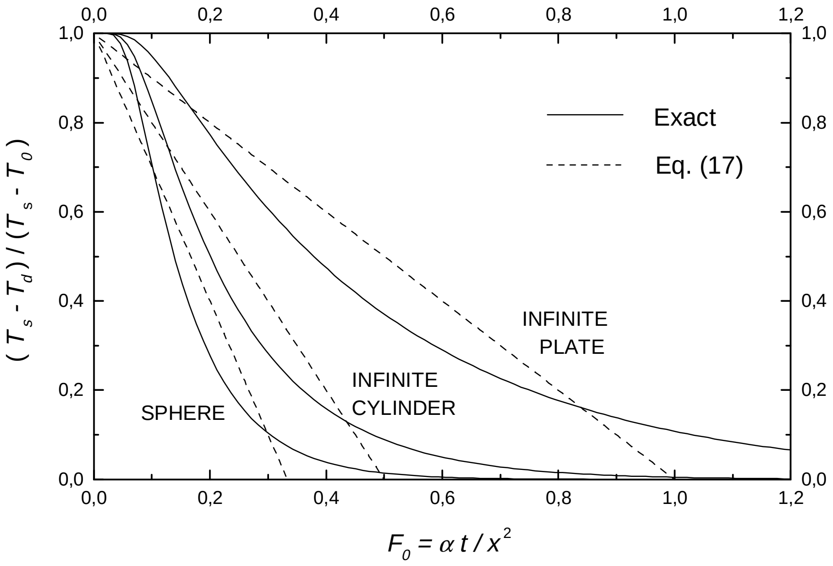

Figure 2. Exact (solid lines) and approximate (dashed lines) thermal evolution at the centre of several objects when immersed in an isothermal bath at temperature T_s. The quantity x is the distance from the centre to the surface (i.e. the half-thickness for a plate, but the radius for a cylinder or a sphere), T_0 is the initial temperature of the object, and $\alpha \equiv \kappa/c_e\rho$, where κ is the heat conductivity, c_e is the specific heat and ρ the density of the material.

Here ΔU is the change of internal energy ($U = uV$, where V is the volume) needed to reach the temperature T_d at the centre of the plate, and $\overline{J}_q$ is the mean heat flux that enters through both plate boundaries, of surface area A. We note from figure 3 that when T_d is reached at the centre of the plate, the temperature is higher elsewhere. Thus we can replace equation (5) by the approximate expression

$$\Delta U \approx V\rho c_e(T_d - T_0). \qquad (12)$$

Note that this equation is an approximation, because only the centre of the plate has temperature T_d at time t_d (see figure 3). Concerning the value of J_q at the plate surface, this comes from Fourier's law (6):

$$J_q = \kappa \left| \frac{\partial T}{\partial x} \right|_{\text{plate surface}}. \qquad (13)$$

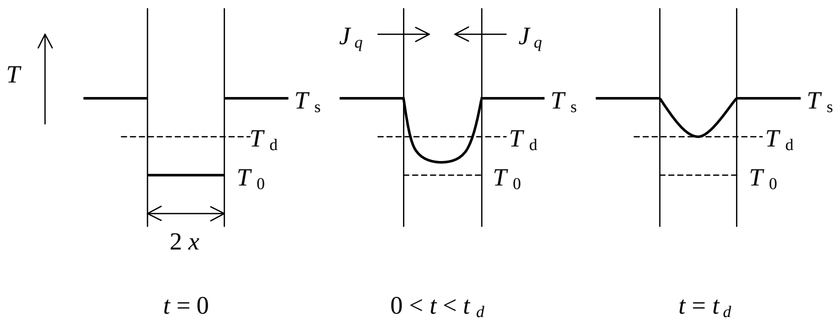

Figure 3. Schematic evolution of the temperature profile in a plate.

So, it seems reasonable to consider $\overline{J}_q$ as being proportional to the initial temperature step $T_s - T_0$ and inversely proportional to the half-width of the plate, x (see figure 3). So, we can write

$$\bar{J}_q \approx \kappa \frac{T_s - T_0}{x}. \tag{14}$$

Of course, a proportionality constant different from unity could be included in the approximate equations (12) and (14). However, its convenience as well as the correctness of the hypothesis applied in these equations will ultimately be tested by the solution to which they lead. Substitution of equations (14) and (12) in equation (11) yields an approximate value for t_d:

$$t_d \approx \frac{V}{2A} \frac{\rho c_e}{\kappa} x \frac{T_d - T_0}{T_s - T_0} = \frac{1}{\alpha} x^2 \frac{T_d - T_0}{T_s - T_0} \tag{15}$$

where we have made use of the definition of the thermal diffusivity (equation (8)) and the fact that $V/A = 2x$ for a plate (figure 3). Finally, by rearranging equation (15) we obtain

$$\frac{T_s - T_d}{T_s - T_0} \approx 1 - \frac{\alpha t_d}{x^2} \equiv 1 - F_0. \tag{16}$$

In spite of the simplicity of this derivation, the result for t_d deviates by less than 20% from the exact solution, except for the asymptotic behaviour at long times and the initial transient (see figure 2). These greater discrepancies can be understood as follows. Our approximate solution considers that the heat flux is constant (equation (14)), which results in a constant rate of temperature diminution (equation (16)). However, since the temperature is uniform inside the plate at $t = 0$ and $t = \infty$, no heat arrives at the centre and the rate of temperature change is, consequently, zero at these times. Now, one may generalize equation (16) to other geometries:

$$\frac{T_s - T_d}{T_s - T_0} \approx 1 - f_A F_0 \tag{17}$$

where f_A is a geometrical factor accounting mainly for changes in the ratio A/V when passing from an infinite plate to an infinite cylinder or to a sphere. We obtain $f_A = 1$, 2 and 3 for an infinite plate, an infinite cylinder and a sphere, respectively. This factor has been calculated from the area per unit volume in every case (for a cylinder and a sphere, the quantity x appearing in $F_0 \equiv \alpha t_d / x^2$ is the radius).

3. Boiling an egg

We will now apply equation (17) to our problem by considering the egg as spherical (mean radius $x = 2.3$ cm). The main difficulty is discovering the thermal parameters of the egg, namely: (a) at what temperature the white and the yolk cook and (b) what their thermal conductivities (κ) and specific heats (c_e) are. The first question can be easily solved experimentally by cooking a small amount of egg (e.g., at the bottom of a test tube) in a water bath whose temperature is raised steadily. By doing so one discovers that it cooks at 70 °C (T_d). As regards κ and c_e, these parameters are not currently available unless from specialized sources. Their measurement is also complicated. Thus, we propose the following simplification: since the water content in the egg is very high (this holds for most living matter)†, we will take the thermal parameters of water ($\kappa = 0.64$ J s^{-1} m^{-1} K^{-1}, $c_e = 4.18 \times 10^3$ J kg^{-1} K^{-1}, $\rho = 1000$ kg m^{-3})‡. Of course, if there were purely water,

† The water content is about 90% in the white and about 50% in the yolk.

‡ The real values for the egg are very similar. We performed a simple experiment in order to measure them. The results were $\alpha_{\text{white}} = 1.5\alpha_{\text{water}}$ and $\alpha_{\text{yolk}} = 1.1\alpha_{\text{water}}$, with minor variations once cooked. One effect that is not accounted for is the heat absorbed during the process of cooking that corresponds to the irreversible endothermic transformation which changes the structure and appearance of the white and yolk. The heat absorbed per gramme of protein is very small (about 1.7 J) and does not influence the temperature evolution significantly.

equation (14) would not apply because of the convective currents that would develop inside the egg. However, proteins almost immobilize the water, making convection impossible.

Finally we take our egg out of the refrigerator ($T_0 = 4$ °C) and immerse it in boiling water ($T_s = 100$ °C). According to equation (17) the time needed to cook it completely (i.e. hard-boiled) will be 13 min. At this point we think that if you were asked the question in the title, your answer would have not been very far from our solution.

4. Summary and perspectives

The heating of simple geometric objects dipped into an isothermal bath has been analysed under simple approximations in order to estimate the time evolution of the temperature. Our method succeeds in identifying an important dimensionless parameter, the Fourier modulus (equation (10)), which summarizes the effect of both the geometry and the thermal parameters on the heating time. Namely, the time needed to reach a given temperature is proportional to square of the linear dimension of the object and inversely proportional to the heat diffusivity of the material. From the quantitative point of view, our approximate solution approaches the exact thermal evolution with reasonable accuracy. In any case, we think that the most relevant contribution of this paper is to provide a simple derivation of the general relationship (equation (2)) based on a qualitative analysis of Fourier's and Fick's laws (equation (1)). We have shown (equation (17)) that thermalization is achieved when the Fourier modulus approaches unity. This condition leads to

$$t_d \approx \frac{x^2}{\alpha}. \tag{18}$$

Our simple method makes it possible to analyse non-steady problems that would otherwise be intractable in introductory courses in physics. For instance, the heating or cooling of homogeneous objects can be measured and the results compared to our approximate solution (equation (17)) (such an experiment is described in [6]). Also, the scale rules contained in the Fourier modulus can be easily assessed without the need to know the material parameters. The thermal diffusivity can be obtained, etc. These experiments would extend the teaching possibilities for diffusion and heat conduction, which are usually restricted to time-independent problems.

Of course, the reader is also invited to pay greater attention to the time usually needed to boil an egg.

Acknowledgment

The authors are indebted to C Carretero for the information she provided concerning the structure and thermal behaviour of an egg.

References

[1] Kane J W and Sternheim M M 1984 *Physics* (New York: Wiley) ch 11
[2] Berry B S 1983 Atomic diffusion and the breakdown of Hooke's law *Phys. Teach.* **21** 435
[3] See, e.g., deGroot S R and Mazur P 1984 *Non-Equilibrium Thermodynamics* (New York: Dover)
[4] Welty J R, Wicks C E and Wilson R E 1984 *Fundamentals of Momentum, Heat and Mass Transfer* (New York: Wiley) ch 18
[5] Isachenko V, Osipova V and Sukomel A 1980 *Heat Transfer* (Moscow: Mir) ch 3
[6] Unsworth J and Duarte F J 1979 Heat diffusion in a solid sphere and Fourier theory: an elementary practical example *Am. J. Phys.* **47** 981–3