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Radiative transfer in the framework of extended irreversible thermodynamics

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Abstract. The thermodynamics of some non-equilibrium systems composed of matter and radiation is analysed. It is shown that a simple theory based on extended irreversible thermodynamics (EIT) provides a formulation which is consistent with the second law for situations in which the time variation of the radiative flux cannot be neglected. The theory also ensures thermodynamic stability. The results are applied to make some estimations of the generalized temperature in the stars.

1. Introduction

Classical irreversible thermodynamics (CIT), also called local-equilibrium thermodynamics, is a very useful theory which has been applied to a great variety of problems (De Groot and Mazur 1984). Its range of validity is restricted to situations in which thermodynamic fluxes are linear functions of the gradients of the state variables (Prigogine 1949, De Groot and Mazur 1984). A well known example is Fourier's law of heat conduction,

$$\boldsymbol{q} = -\lambda_q \boldsymbol{\nabla} T \tag{1}$$

where $\boldsymbol{q}$ is the heat flux, λ_q stands for the thermal conductivity and T is the absolute temperature.

The need to consider more general situations has been pointed out many times, both on the basis of conceptual (Maxwell 1867, Cattaneo 1948, Vernotte 1958) and experimental (Joseph and Preziosi 1989, 1990, Lebon and Cloot 1989, Dreyer and Struchtrup 1993) motivations. For situations in which, for example, equation (1) is replaced by the Maxwell–Cattaneo equation,

$$\tau_q \frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} + \boldsymbol{q} = -\lambda_q \boldsymbol{\nabla} T \tag{2}$$

where τ_q is the relaxation time and t is the time, a consistent formulation is provided by extended irreversible thermodynamics (EIT) (Jou *et al* 1988, 1993, Müller and Ruggeri 1992).

Whereas many detailed studies based on CIT and EIT have dealt with such phenomena as conductive and convective heat transfer, the thermodynamics of radiative transfer has been less thoroughly analysed. The present study is motivated by two previous results:

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(i) *A very anisotropic radiation field.* Ciano and Verhás (1990, 1991) considered one-dimensional radiative transfer: in a region of three-dimensional space, all photons are assumed to move along the x-direction (forward and backward) and the matter is assumed to be macroscopically at rest. Under the additional assumption that the radiative energy density u_r depends only on the temperature, they showed that

$$\tau \frac{dF}{dt} + F = -\lambda_1 \frac{\partial T}{\partial x} \tag{3}$$

with

$$\tau = \frac{1}{\sigma_a c} \qquad \text{and} \qquad \lambda_1 = \frac{c}{\sigma_a} \frac{du_r}{dT} \tag{4}$$

and the radiative energy flux is $\boldsymbol{F} = (F, 0, 0)$, σ_a stands for the absorption coefficient and c is the speed of light. We mention that equations (3) and (4) can also be easily derived from the radiative transfer equation.

Equation (3) is a radiative analogue of the heat conduction equation (2). This suggests that the thermodynamics of such a system may be described by means of the EIT theory. This is analysed in sections 2 and 3.

(ii) *An almost isotropic radiation field.* An important equation in the theory of stellar interiors is the following (Schwarzschild 1965, Chandrasekhar 1967, Bowers and Deeming 1984):

$$\boldsymbol{F} = -\lambda_2 \boldsymbol{\nabla} T \tag{5}$$

with

$$\lambda_2 = \frac{4acT^3}{3\sigma_a} \tag{6}$$

where a is the blackbody constant. If σ_a is frequency-dependent then the Rosseland mean appears in (6) instead of σ_a, but this is not necessary to stress the fundamental thermodynamical features. Equation (5), which is also derived in equilibrium diffusion theory of radiative transfer (Pomraning 1973), is a radiative analogue of the Fourier heat conduction equation (1). This suggests that (5) may be included in the CIT theory (see section 2). In the derivation of (5) it is assumed that $\boldsymbol{F}$ does not appreciably depend on time. This suggests that in situations for which $\boldsymbol{F}$ depends on time, (5) may be generalized into an equation of the type of (3) and included in the framework of EIT. This point is developed in section 3.

2. General theory

Let e and $\boldsymbol{J}_e$ stands for the total energy per unit mass and the total energy flux of an arbitrary system. The energy conservation law reads

$$\frac{\partial}{\partial t}(\rho e) = -\boldsymbol{\nabla} \cdot \boldsymbol{J}_e \tag{7}$$

where ρ is the matter density. If the system is formed by matter macroscopically at rest (i.e. with vanishing barycentric velocity, $\boldsymbol{v} = 0$) and radiation, and energy is transported solely by means of radiation we have

$$e = u_m + \frac{u_r}{\rho} \equiv u \tag{8}$$

$$\boldsymbol{J}_e = \boldsymbol{F} \tag{9}$$

where u_m is the internal energy of matter per unit mass. Taking into account that the mass balance equation implies that $\partial\rho/\partial t = 0$, (7) may be written as

$$\rho\frac{\mathrm{d}u}{\mathrm{d}t} = -\nabla\cdot\boldsymbol{F} \tag{10}$$

where the general definition of the total derivative is $\mathrm{d}u/\mathrm{d}t = \partial u/\partial t + \boldsymbol{v}\cdot\nabla u$, which in our case ($\boldsymbol{v} = 0$) reduces to $\mathrm{d}u/\mathrm{d}t = \partial u/\partial t$. For $\boldsymbol{v} \neq 0$, additional terms in equation (10) have to be taken into account (Pomraning 1973, Mihalas 1986), but the thermodynamics of such systems will not be considered here.

The classical approach to the thermodynamics of a system formed by matter and radiation out of equilibrium introduces a temperature related to the matter and a radiation temperature which depends on frequency and direction (Planck 1959, Callies and Herbert 1988, Albert 1991). On the other hand, in Eu and Mao (1992) and Mao and Eu (1993) it has been recently argued for the use of a single temperature, essentially on the basis that a thermometer placed in a radiation–matter system only measures a single temperature (they note that the classical procedure mentioned above is analogous to the sometimes used approach to multicomponent systems that introduces different temperatures for different species, such as in Ichimaru (1973)). Since only a temperature field appears in equations (3) and (5) it seems to us difficult for the classical approach to lead to a simple thermodynamic theory that includes such equations. With this perspective we will for the purposes of the present paper write down the local-equilibrium Gibbs equation for a system formed by radiation and a single species of matter with constant density in the simple form

$$T\frac{\mathrm{d}s_{\mathrm{le}}}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}t} \tag{11}$$

where s_{le} is the local-equilibrium entropy of the radiation–matter system per unit mass of matter. A detailed discussion on Gibbs and more general equations for radiation–matter systems on the basis of kinetic theory can be found in Eu and Mao (1992).

Use of (10) into (11) gives

$$\rho\frac{\mathrm{d}s_{\mathrm{le}}}{\mathrm{d}t} = -\frac{1}{T}\nabla\cdot\boldsymbol{F}. \tag{12}$$

Let $\boldsymbol{J}^s_{\mathrm{le}}$ and σ^s_{le} stand for the local entropy flux and the local entropy rate of production, respectively. The entropy balance law is sought in the usual form

$$\rho\frac{\mathrm{d}s_{\mathrm{le}}}{\mathrm{d}t} = -\nabla\cdot\boldsymbol{J}^s_{\mathrm{le}} + \sigma^s_{\mathrm{le}}. \tag{13}$$

Comparison of (13) with (12) leads to the identifications

$$\boldsymbol{J}^s_{\mathrm{le}} = \frac{1}{T}\boldsymbol{F} \qquad \text{and} \qquad \sigma^s_{\mathrm{le}} = \boldsymbol{F}\cdot\nabla\left(\frac{1}{T}\right). \tag{14}$$

In the case when radiative transfer can be described by a Fourier-type equation,

$$\boldsymbol{F} = -\lambda\nabla T \tag{15}$$

with $\lambda \geqslant 0$ (see, e.g., equations (5) and (6)), then (14) ensures the validity of the second law,

$$\sigma^s_{\mathrm{le}} = \frac{\lambda}{T^2}(\nabla T)^2 \geqslant 0. \tag{16}$$

However if, instead of (15), an equation of the Maxwell–Cattaneo type (such as (3)) holds, i.e.

$$\tau\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}t} + \boldsymbol{F} = -\lambda\nabla T \tag{17}$$

then the local-equilibrium hypothesis (11) clearly breaks down since (16) does not hold and the semi-positiveness of σ^s is no longer ensured. In order to avoid this contradiction with the second law, one clearly needs a more general thermodynamic theory that makes the second law compatible with equation (17). We want to see whether the assumptions and formalism of EIT can be successfully applied to this case. Therefore we follow the EIT procedure and assume that the entropy does not depend only on u, as in (11), but also on the radiative energy flux $\boldsymbol{F}$. Then

$$\mathrm{d}s = \theta^{-1}\,\mathrm{d}u - T^{-1} v \boldsymbol{\alpha} \cdot \mathrm{d}\boldsymbol{F} \tag{18}$$

where in analogy with (11) we have introduced a generalized temperature θ, and defined $\boldsymbol{\alpha}$ through

$$\theta^{-1}(u, \boldsymbol{F}) = \left(\frac{\partial s}{\partial u}\right)_{\boldsymbol{F}} \qquad \text{and} \qquad T^{-1} v \boldsymbol{\alpha}(u, \boldsymbol{F}) = -\left(\frac{\partial s}{\partial \boldsymbol{F}}\right)_u \tag{19}$$

where $v = 1/\rho$. For simplicity we restrict the theory to small values of the radiative flux by assuming the linear relation

$$\boldsymbol{\alpha}(u, \boldsymbol{F}) = \alpha(u)\boldsymbol{F}. \tag{20}$$

Following the same arguments as in the EIT theory of heat conduction (Jou *et al* 1988, 1993) one finds that the second law, (10), (17), (18) and (20) lead to

$$\alpha = \frac{\tau}{\lambda T} \tag{21}$$

where τ is, as in (17), a parameter with time dimensions, identified with the relaxation time of the radiative energy flux, and

$$\theta^{-1} = T^{-1} - \frac{\partial}{\partial u}\left(\frac{\tau v}{\lambda T^2}\right) \frac{\boldsymbol{F} \cdot \boldsymbol{F}}{2}. \tag{22}$$

According to (18), the generalized entropy is given by

$$s(u, \boldsymbol{F}) = s_{\mathrm{le}}(u) - \frac{\tau v}{2\lambda T^2} \boldsymbol{F} \cdot \boldsymbol{F} \tag{23}$$

where we can see that the non-equilibrium entropy differs from the local-equilibrium entropy in the term that depends on the radiative energy flux and its relaxation time. In the following sections we will explore the consequences of this formalism in special systems.

3. Special cases

(i) *A very anisotropic radiation field.* For the case (i) explained in the introduction we have, according to (4) and (23),

$$s(u, \boldsymbol{F}) = s_{\mathrm{le}}(u) - \frac{v}{2c^2 T^2} \frac{1}{(\mathrm{d}u_{\mathrm{r}}/\mathrm{d}T)} \boldsymbol{F} \cdot \boldsymbol{F} \tag{24}$$

which is in accordance with the results obtained by Ciano and Verhás (1990, 1991). They based their analysis on Gyarmati's wave approach (Gyarmati 1977), so that in this one-dimensional radiative transfer situation such an approach is equivalent to EIT, obtaining for their result a wider theoretical framework (there has been some controversy on whether EIT and Gyarmati's wave approach are equivalent or not in general, see García-Colín and Rodriguez 1989, Márkus and Gambar 1989, García-Colín and Uribe 1991).

In the present approach, according to (22) and (4) the generalized temperature θ is related to the local-equilibrium temperature T through

$$\theta^{-1} = T^{-1} - \frac{\partial}{\partial u}\left(\frac{v}{c^2T^2}\frac{1}{(\mathrm{d}u_\mathrm{r}/\mathrm{d}T)}\right)\frac{\boldsymbol{F}\cdot\boldsymbol{F}}{2}.$$

(ii) *More general radiation fields.* The utility of EIT is not restricted to the previous very special case. We now show that equation (17) also arises in three-dimensional radiative transfer situations.

For an arbitrary distribution of radiation we have (Pomraning 1973, Chandrasekhar 1967)

$$u_\mathrm{r}(\boldsymbol{r},t) = \frac{1}{c}\int_0^\infty \mathrm{d}\nu \int_{4\pi} \mathrm{d}\Omega\, I_\nu(\boldsymbol{r},\boldsymbol{\Omega},t) \tag{25}$$

$$\boldsymbol{F}(\boldsymbol{r},t) = \int_0^\infty \mathrm{d}\nu \int_{4\pi} \mathrm{d}\Omega\, \boldsymbol{\Omega} I_\nu(\boldsymbol{r},\boldsymbol{\Omega},t) \tag{26}$$

$$\mathsf{P}_\mathrm{r}(\boldsymbol{r},t) = \frac{1}{c}\int_0^\infty \mathrm{d}\nu \int_{4\pi} \mathrm{d}\Omega\, \boldsymbol{\Omega\Omega} I_\nu(\boldsymbol{r},\boldsymbol{\Omega},t) \tag{27}$$

where P_r is the radiation pressure tensor, $\boldsymbol{\Omega}$ a unit vector and $(\boldsymbol{\Omega\Omega})_{\alpha\beta} = \Omega_\alpha\Omega_\beta$, with $\alpha,\beta = x, y, z$. The intensity $I_\nu(\boldsymbol{r},\boldsymbol{\Omega},t)$ is the energy of the photons with frequency in an interval $\mathrm{d}\nu$ (centred at ν) and direction of motion in a solid angle $\mathrm{d}\Omega$ (centred at the direction of $\boldsymbol{\Omega}$) that cross during a time interval $\mathrm{d}t$ (centred at t) a surface $\mathrm{d}\sigma$ (centred at $\boldsymbol{r}$) orthogonal to $\boldsymbol{\Omega}$, divided by $(\mathrm{d}\nu\,\mathrm{d}\Omega\,\mathrm{d}t\,\mathrm{d}\sigma)$.

If, for simplicity, we neglect scattering and induced processes, the equation of radiative transfer reads (Pomraning 1973)

$$\frac{1}{c}\frac{\partial I_\nu(\boldsymbol{r},\boldsymbol{\Omega},t)}{\partial t} + \boldsymbol{\Omega}\cdot\boldsymbol{\nabla} I_\nu(\boldsymbol{r},\boldsymbol{\Omega},t) = -\sigma_\mathrm{a} I_\nu(\boldsymbol{r},\boldsymbol{\Omega},t) + \sigma_\mathrm{e}(\nu) \tag{28}$$

where again for simplicity the absorption coefficient σ_a and the volume emissivity $\sigma_\mathrm{e}(\nu)$ are assumed constant and uniform and we have also applied the gray (or one group) approximation so that σ_a is assumed to be independent of frequency.

Multiplication of (28) by $\boldsymbol{\Omega}$ and integration over all frequencies and over all solid angles gives, making use of (26) and (27),

$$\frac{1}{c}\frac{\partial \boldsymbol{F}(\boldsymbol{r},\boldsymbol{\Omega},t)}{\partial t} + c\boldsymbol{\nabla}\cdot\mathsf{P}_\mathrm{r}(\boldsymbol{r},t) = -\sigma_\mathrm{a}\boldsymbol{F}(\boldsymbol{r},\boldsymbol{\Omega},t) \tag{29}$$

with $[\boldsymbol{\nabla}\cdot\mathsf{P}_\mathrm{r}]_\alpha = \sum_{\beta=1}^{3}\frac{\partial P_{\beta\alpha_\mathrm{r}}}{\partial x_\beta}$. This is still not an equation of the Maxwell–Cattaneo type (17). However, since the matter is by assumption macroscopically at rest, we have, as in (10), $\partial\boldsymbol{F}/\partial t = \mathrm{d}\boldsymbol{F}/\mathrm{d}t$ (the present thermodynamical notation, namely $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \boldsymbol{v}\cdot\boldsymbol{\nabla}$, which has been introduced in (10) and which is also used in radiation hydrodynamics (Pomraning 1973), should not be confused with the notation $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \boldsymbol{c}\cdot\boldsymbol{\nabla}$, with $\boldsymbol{c} = c\boldsymbol{\Omega}$, which is sometimes used in order to write down the radiative transfer equation in a more compact form, as in Callies and Herbert (1988)). Moreover, if we restrict ourselves to radiation fields for which P_r satisfies the following requirements: (i) it is a diagonal tensor, (ii) $P_{xx_\mathrm{r}} = P_{yy_\mathrm{r}} = P_{zz_\mathrm{r}} \equiv p_\mathrm{r}$ and (iii) it depends only on T, then (29) becomes equation (17) with

$$\tau = \frac{1}{\sigma_\mathrm{a}c} \qquad \text{and} \qquad \lambda = \frac{c}{\sigma_\mathrm{a}}\frac{\mathrm{d}p_\mathrm{r}}{\mathrm{d}T}. \tag{30}$$

As an example to illustrate these results we will now consider a case which is somehow opposite to that dealt with in subsection (i): the almost isotropic radiation field mentioned as case (ii) in the introduction. As we have mentioned there, an hypothesis which is made in the derivation of equation (5) is that $\boldsymbol{F}$ does not depend on time. We will derive here a generalization of (5) in non-steady state situations and make a thermodynamic study of the corresponding matter–radiation system in terms of EIT.

As in equilibrium diffusion theory (Pomraning 1973), we restrict ourselves to situations which satisfy the two following conditions.

(1) The Eddington (or classical diffusion) approximation. It is assumed that the intensity of an almost isotropic radiation field can be split into two terms,

$$I_\nu(\boldsymbol{r}, \boldsymbol{\Omega}, t) = \frac{1}{4\pi} I_{0\nu}(\boldsymbol{r}, t) + \frac{3}{4\pi} \boldsymbol{\Omega} \cdot \boldsymbol{I}_{1\nu}(\boldsymbol{r}, t). \tag{31}$$

The second term in (31) is a first-order anisotropic correction to the isotropic term.

(2) We also assume the isotropic term to be locally Planckian,

$$\frac{1}{4\pi} I_{0\nu}(\boldsymbol{r}, t) = \frac{2h\nu^3}{c^2} (\mathrm{e}^{h\nu/kT(\boldsymbol{r},t)} - 1)^{-1} \tag{32}$$

where h is the Planck constant and k is the Boltzmann constant.

As is well known, multiplication of (31) by 1, $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega\Omega}$ and integration over all frequencies and all solid angles yields, respectively, using (25)–(27) and (32),

$$u_\mathrm{r}(\boldsymbol{r}, t) = aT^4(\boldsymbol{r}, t) \tag{33}$$

$$\boldsymbol{F}(\boldsymbol{r}, t) = \int_0^\infty \mathrm{d}\nu\, \boldsymbol{I}_{1\nu}(\boldsymbol{r}, t) \tag{34}$$

$$\mathsf{P}_\mathrm{r}(\boldsymbol{r}, t) = \tfrac{1}{3} u_\mathrm{r}(\boldsymbol{r}, t)\mathsf{U} \tag{35}$$

with $a = 8\pi^5 k^4/15c^3h^3$ the blackbody constant and U the identity matrix. For the special case of blackbody radiation, matter and radiation are in complete thermodynamic equilibrium so that T is homogeneous and stationary and the radiation field is isotropic ($\boldsymbol{I}_{1\nu} = 0$), thus no neat heat transfer takes place (i.e. $\boldsymbol{F} = 0$).

From equations (35) and (33) it follows immediately that the three conditions that P must satisfy for equations (30) to be applicable are fulfilled. Therefore the non-steady state equation corresponding to the Fourier (or Fick) type equation (5) is precisely the Maxwell–Cattaneo-type equation (17). Moreover, according to (30), (33) and (35) we have

$$\tau = \frac{1}{\sigma_\mathrm{a} c} \qquad \text{and} \qquad \lambda = \frac{c}{3\sigma_\mathrm{a}} \frac{\mathrm{d}u_\mathrm{r}}{\mathrm{d}T} = \frac{4caT^3}{3\sigma_\mathrm{a}}. \tag{36}$$

Comparison of (36) with (4) shows that the relaxation time is the same as in the highly anisotropic case dealt with before, whereas the thermal conductivity is one-third of the result for that one-dimensional heat transfer situation. On the other hand, the thermal conductivity λ in (36) is the same as λ_2 in (6) as it should, since in the present situation (17) must reduce to (5) in the steady state. The Fourier-type equation (5) may be applied assuming the radiation flux to vary very slowly in time (see Schwarzschild 1965); otherwise one may use equation (17). Equations of this kind may be of interest in several astrophysical phenomena (Schweizer 1985a, b, Uddey and Israel 1982) and, in contrast with CIT, EIT gives an adequate thermodynamical description under the assumptions used here. Then we have from equations (36) and (23)

$$\lambda = \tfrac{4}{3} ac^2 \tau T^3 \tag{37}$$

$$s(u, \boldsymbol{F}) = s_\mathrm{le}(u) - \frac{3v}{8ac^2T^5} \boldsymbol{F} \cdot \boldsymbol{F}. \tag{38}$$

Equation (37) has been obtained before in a relativistic framework through macroscopic (Weinberg 1971), kinetic (Uddey and Israel 1982) and fluctuation (Pavón *et al* 1983) theories. The simplicity of the present approach is due to the assumption that the matter is macroscopically at rest. Guided by the analogies with heat conduction, we have found some results that may be applied to a variety of radiation fields.

Whereas CIT may be used in some situations such that $\boldsymbol{F}$ does not depend on time (equation (15)), an adequate description outside the steady state (equation (17)) is provided by EIT. Assuming a radiative flux of the form $\boldsymbol{F} = \boldsymbol{F}_0 \cos(\omega t + \boldsymbol{k} \cdot \boldsymbol{x} + \alpha)$, (17) and (15) show that the CIT description is adequate for $\omega\tau \ll 1$, but not for fast phenomena and small values of the absorption coefficient (see equations (4) or (36)). Our conclusion is that an adequate treatment is then provided by EIT.

4. Stability

The requirement of stability has been used in non-radiative EIT theories in order to establish the admissible values of the heat flux (Jou *et al* 1993, Criado-Sancho and Llebot 1993). The local-equilibrium state is stable provided that the entropy is a convex function of the extended variables. According to (23), these variables are u and F for our radiative EIT model. Therefore stability is ensured if the following three conditions hold:

$$\begin{cases} \partial^2 s/\partial u^2 < 0 \\ \partial^2 s/\partial F^2 < 0 \\ (\partial^2 s/\partial u^2)(\partial^2 s/\partial F^2) - (\partial^2 s/\partial u \partial F)^2 > 0. \end{cases} \tag{39}$$

These conditions can be explicitly calculated for the non-stationary extension of equilibrium diffusion theory dealt with in subsection (ii) of section 3. Assuming for simplicity that the matter content of the system consists of an ideal monatomic gas, so that (8) and (33) yield $u = 3kT/2m + aT^4/\rho$, with m the molecular mass, the first derivatives of (38) are

$$\begin{cases} \dfrac{\partial s}{\partial u} = \dfrac{1}{T}\left(1 + \dfrac{15F^2}{8c^2 u_\mathrm{r}(u_\mathrm{m}\rho + 4u_\mathrm{r})}\right) \\ \dfrac{\partial s}{\partial F} = -\dfrac{3F}{4c^2 T u_\mathrm{r}\rho}. \end{cases} \tag{40}$$

Finding out the second derivatives, the stability conditions (39) read, respectively,

$$\begin{cases} -\dfrac{1}{T(u_\mathrm{m} + 4u_\mathrm{r}/\rho)}\left(1 + \dfrac{45(u_\mathrm{m}\rho + 6u_\mathrm{r})}{4c^2 u_\mathrm{r}(u_\mathrm{m}\rho + 4u_\mathrm{r})^2} F^2\right) < 0 \\ -\dfrac{3}{4c^2 T u_\mathrm{r}\rho} < 0 \\ F^2 < c^2 u_\mathrm{r}^2 f(x) \end{cases} \tag{41}$$

with $f(x) = 2(x+4)^2/(15(x+1))$ and $x = \rho u_\mathrm{m}/u_\mathrm{r}$. It is clear that the two first conditions are always satisfied. On the other hand, it is well known that radiative transfer is an intrinsically flux-limited theory: (25) and (26) imply that $F \leqslant c u_\mathrm{r}$. Now it is easy to see that $f(x) > 1$ for any value of $x > 0$. Therefore the third condition of stability will be satisfied as well in any real situation. We conclude that the EIT radiative transfer entropy (38) implies stability for any possible values of the thermodynamical variables u and $\boldsymbol{F}$.

5. Some estimations of the generalized temperature in the stars

Some experiments have been proposed in which the EIT non-equilibrium temperature θ could be tested (Jou and Casas-Vázquez 1992, Casas-Vázquez and Jou 1994). It is therefore of interest to find out experimental situations in which the difference between T and θ becomes appreciable. For some heat conduction situations this difference may be of about 0.4%. Note that, according to (22), both temperatures are equivalent either in the absence of radiative flux or for vanishing relaxation time.

Casas-Vázquez and Jou (1994) have provided a rough estimation of the difference between T and θ at the surface of a star. Here we would like to make some estimations making use of the present more detailed model. In most stars, heat conduction is negligible but radiative transfer has to be taken into account. The approximation (31) is currently used (Schwarzschild 1965, Stix 1989) and instead of (8) we have

$$e = u_{\mathrm{m}} + \frac{u_{\mathrm{r}}}{\rho} + \psi \equiv u + \psi \tag{42}$$

with ψ the gravitational energy per unit mass. Equations (37) and (22) yield

$$\theta^{-1} = T^{-1} - \frac{\partial}{\partial u}\left(\frac{3v}{4ac^2T^5}\right)\frac{\boldsymbol{F}\cdot\boldsymbol{F}}{2}. \tag{43}$$

The internal energy is approximately given by a relationship of the monatomic ideal gas type (Schwarzschild 1965, Stix 1989),

$$u_{\mathrm{m}} = \frac{3kT}{2m_{\mathrm{H}}\mu} \tag{44}$$

where m_{H} stands for the mass of a proton and μ (the mean molecular mass measured in proton masses) takes into account the chemical composition and the degrees of ionization. Insertion of (42), (33) and (44) into (43) yields

$$\theta^{-1} = T^{-1} + \frac{1}{(3k\rho/2m_{\mathrm{H}}\mu) + 4aT^3}\,\frac{15}{8ac^2T^6}F^2. \tag{45}$$

When the surface of a star is considered, it is important to keep in mind that since the matter density does *not* vanish outside the surface (see Stix 1989), any given area on the surface is crossed by some photons that have been emitted in outer layers (otherwise, the inward radiative flux would vanish on the surface and the radiation field would be strongly anisotropic, so that the approximation (31) would break down).

It seems reasonable to begin making an estimation of (45) for the Sun since this is the star for which more observational data are available. In order to make an estimation at the solar surface we use the following values (from the solar model in Stix (1989), p 47): $T = 5778$ K, $\rho = 3.03 \times 10^{-4}$ kg m^{-3}, $\mu = 1.251$ and $F = 6.31 \times 10^7$ W m^{-2}. Taking into account that $k/m_{\mathrm{H}} = 8254.45$ J K^{-1} kg^{-1}, (45) gives $(T-\theta)/T$ of the order 10^{-6}, i.e. a modification which is only of the order of 10^{-4}%. Such a small modification is of no practical interest at all but may be useful in order to compare the present approach with that of Casas-Vázquez and Jou (1994), which yields a modification of about 3% at any stellar surface. The present estimation relies heavily on the introduction of a single temperature for the radiation–matter system in the Gibbs equation (11) and its generalization (18), which has the effect that u in equation (43) includes, according to (42), both the matter internal energy and the radiation energy, i.e. $u = u_{\mathrm{m}} + u_{\mathrm{r}}/\rho$ (this is the main difference with both the conductive case, for which $u = u_{\mathrm{m}}$ (see, e.g., Jou *et al* (1993), and with the approach of Casas-Vázquez and Jou (1994) to the radiative case, in which they take $u = u_{\mathrm{r}}/\rho$). For situations in which $u_{\mathrm{m}} \ll u_{\mathrm{r}}/\rho$, (43) and (45) reduce to the result previously obtained by

Casas-Vázquez and Jou (1994). In other words, the difference between the present result and that of Casas-Vázquez and Jou (1994) is due to the fact that, in order to obtain a rough estimation, these authors did not take into account the effect of the matter content of the system on the EIT temperature. The present model does not contradict their results but shows in which limit they may be applied. Indeed, whereas for the former numerical values of the Sun (33) and (44) show that at the solar surface we have $u_{\mathrm{m}} \gg u_{\mathrm{r}}/\rho$ (which explains the difference between the present result and that of Casas-Vázquez and Jou (1994)), there are many stars other than the Sun for which this condition does not hold and the difference between T and θ may be appreciable. To see this we may refer to the stellar models provided by Kurucz (1979). These are classical models of the atmosphere and shallow interior of stars. All the models in Kurucz (1979) are based on the same well known equations. In these equations, Kurucz substitutes specific values of different stellar parameters (such as the effective temperature). Among others, he obtains models for the Sun and Vega. He uses these two special cases to check that the prediction of the models agree very well with observations. It is easily checked that many of the other models provided by Kurucz yield a difference between T and θ similar to that obtained by Casas-Vázquez and Jou (1994). For example, for the first stellar model on p 55 of Kurucz (1979) we have $T = 7928$ K, $\rho = 3.07 \times 10^{-8}$ kg m^{-3}, $\mu = 0.9549$ and $F = 2.32 \times 10^{8}$ W m^{-2}. When these values are used in equations (33) and (44) it is found that u_{m} and u_{r}/ρ are of the same order of magnitude and equation (45) gives $\theta = 7726$ K, so that the correction $(T - \theta)/T$ is of the order of 2.5%, instead of the rough result of 3% evaluated by Casas-Vázquez and Jou (1994).

One may also consider solar regions, other than the surface, in which many interesting processes occur (Stix 1989). Below the solar surface ρ and T increase very rapidly with depth, which combined with the slow variation of F (using the results of the same solar model) gives a difference between T and θ which is even lower than the value we have obtained for the surface. However, a convective model (i.e. $\boldsymbol{v} \neq 0$) may change this result since convection is important in the shallow interior (Stix 1989).

In the lower solar atmosphere F can be taken equal to the surface value, whereas ρ strongly decreases at increasing height and the order of magnitude of T does not change. In view of (45), this will lead to a higher difference between the two temperatures: for the model reproduced in Stix 1989, p 145 ('model C' of Vernazza *et al*) the maximum difference between T and θ is of about 86 K and is reached at a height of about 1280 km, where it is thought that $T = 6220$ K, $\rho = 9.82 \times 10^{-9}$ kg m^{-3} and $\mu = 1.0537$. This gives a correction of about 1.4%. This difference then decreases since the temperature abruptly increases with height as the solar corona is approached. On the other hand it must be mentioned that (31) is a very rough approximation in the atmosphere (Stix 1989, Zirin 1988).

We stress that the previous numerical values are only rough estimations. More realistic models should include, among other effects, heat convection, scattering and the frequency-dependence of the absorption coefficient. In fact a completely consistent estimation of the difference between T and θ would require one to take into account the generalized temperature θ in all the thermodynamic equations of the stellar models. And it should not be forgotten that assumption (32), although currently used, might require some kind of modification in the framework of EIT since it is T and not θ that appears in it. So the results in this section by no means pretend to be definitive but only mean to illustrate the possibility of making comparisons between the local-equilibrium and the EIT temperatures. This can be regarded as a motivation for further work which analyses the possibility of observable consequences of the generalized temperature in radiative transfer situations.

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