



Behaviour of entropy in relaxational hydrodynamics

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Abstract

It is shown that the classical entropy does not increase monotonically for an isolated fluid in which sound waves are attenuated according to a relaxational evolution equation. In contrast, the generalized entropy of extended irreversible thermodynamics does show a monotonic increase and is therefore more suitable for the description of relaxation-time phenomena in fluid systems.

1. Introduction

Classical transport equations such as the Fourier law for heat conduction and the Newton and Stokes laws for a viscous flow imply that the velocity of propagation of the perturbations diverges in the high-frequency limit. This feature can be avoided by taking into account the relaxation times of the fluxes, such as the heat flux, the traceless deviator of the viscous pressure tensor and the bulk viscous pressure. However, the relaxational equations thus obtained, when combined with the local-equilibrium hypothesis no longer imply the validity of the second law of thermodynamics [1,2]. This has been shown in an explicit way by computing the evolution of the entropy of an isolated rigid system where hyperbolic heat conduction occurs [3]. We will try here to show an hydrodynamical analogue of this behaviour, with the intention that the need for a more general thermodynamic theory becomes made completely clear in a variety of situations.

Let P_{ij}^ν stand for the components of the viscous pressure tensor of the fluid. This tensor can be decomposed by defining a bulk viscous pressure p^ν and a traceless deviator \hat{P}_{ij}^ν ,

$$p^\nu = \frac{1}{3}(P_{xx}^\nu + P_{yy}^\nu + P_{zz}^\nu), \quad (1)$$

$$\hat{P}_{ij}^\nu = P_{ij}^\nu - p^\nu \delta_{ij}, \quad (2)$$

where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$.

As an example of relaxational equations, consider the simplest Maxwell–Cattaneo equations [4] for the traceless deviator and the bulk viscous pressure,

$$\tau_2 \frac{d\hat{P}_{ij}^\nu}{dt} + \hat{P}_{ij}^\nu = -2\eta \hat{V}_{ij}^\nu, \quad (3)$$

$$\tau_0 \frac{dp^\nu}{dt} + p^\nu = -\xi \nabla \cdot \mathbf{v}, \quad (4)$$

where η and ξ are the shear and bulk viscosities, respectively, \mathbf{v} is the local fluid velocity, τ_2 and τ_0

are the relaxation times (their subindexes indicate the tensorial order of the corresponding equation) and

$$\mathring{V}_{ij} = \frac{1}{2} \left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) - \frac{1}{3} \delta_{ij} \nabla \cdot \mathbf{v}. \quad (5)$$

The first terms in Eqs. (3) and (4) give a correction to the Newton and Stokes laws for viscous flow, respectively. The full equations (3) and (4) are derived by the Grad thirteen-moment kinetic theory method [5]; in fact if more moments are taken into account, more complicated equations result. Equations of this kind have been applied to many hydrodynamical problems: ultrasound propagation [6], shear waves [7], shock waves [8], rheology [9], etc. (a panoramic view, including experimental results which show the limitations of the Newton–Stokes approach, can be found in Ref. [1]). Here, we would like to address the conceptual implications of equations of this kind from a thermodynamical point of view.

We will for simplicity deal with an ideal monatomic gas, so that in accordance with the kinetic theory of gases [1] we have $p^v = 0$ but $\nabla \cdot \mathbf{v} \neq 0$ in general. We assume η and ξ to be independent of position and time so that (4) implies $\xi = 0$ and Eq. (2) yields $P_{ij}^v = \mathring{P}_{ij}^v$ (it is easy to extend the analysis which will be developed here to the case $p^v \neq 0$, $\xi \neq 0$).

2. Sound

We wish to study the evolution of the entropy of an isolated fluid in which sound waves are attenuated due to the internal friction. Consider a fluid contained in a cylindrical vessel of section A . We take the x -axis normal to the section, and the two plane surfaces of the cylinder to be located at $x = 0$ and $x = L$. All variables are assumed to depend on the x -coordinate and time only and the fluid is assumed to have velocity $\mathbf{v} = (v, 0, 0)$. Thus the non-slip boundary condition is for simplicity not required to hold. A situation in which this is experimentally observed is that of gases at sufficiently low densities so that the velocity varies appreciably over the scale of the mean free path [10] (again, the analysis presented here can be generalized in order

to take the non-slip boundary condition into account). Moreover, we assume the thermal conductivity of the fluid to be negligible so that heat conduction can be ignored (the non-adiabatic case can also be studied by means of an additional Maxwell–Cattaneo equation for the heat flux. However, the behaviour of the entropy associated to such an equation has already been explicitly analyzed [3]).

Consider a perturbation around a reference equilibrium rest state with density ρ_0 , pressure p_0 , temperature T_0 , specific internal energy u_0 and specific entropy s_0 , so that $\rho = \rho_0 + \delta\rho$, $p = p_0 + \delta p$, $\mathbf{v} = (\delta v, 0, 0)$, $\mathring{P}_{ij}^v = P_{ij}^v = \delta P_{ij}^v$ and $s = s_0 + \delta s$.

The linearized balance equations of mass and x -momentum read

$$\frac{\partial}{\partial t} \delta\rho + \rho_0 \frac{\partial}{\partial x} \delta v = 0, \quad (6)$$

$$\rho_0 \frac{\partial}{\partial t} \delta v = - \frac{\partial}{\partial x} \delta p - \frac{\partial}{\partial x} \delta P_{xx}^v. \quad (7)$$

Neglecting heat conduction, the linearized balance equation of internal energy leads to [1]

$$\frac{c_p}{\alpha c^2} \frac{\partial}{\partial t} \delta p + \left(\frac{p_0}{\rho_0} - \frac{c_p}{\alpha} \right) \frac{\partial}{\partial t} \delta\rho = -p_0 \frac{\partial}{\partial x} \delta v,$$

with $c_p = T(\partial s/\partial T)_p$ the specific heat at constant pressure, $\alpha = -\rho^{-1}(\partial\rho/\partial T)_p$ the coefficient of thermal expansion and $c = [(\partial p/\partial\rho)_s]^{1/2}$ the Laplace sound velocity. This equation and (6) yield

$$\frac{\partial}{\partial t} \delta p = c^2 \frac{\partial}{\partial t} \delta\rho. \quad (8)$$

Note that this equation is valid only for adiabatic motion. Also, the fact that an additional term in the entropy perturbation δs does not appear in (8) does not imply that δs vanishes but that the corresponding term in this equation is negligible, as it will be explicitly checked out for the solutions we will deal with (in the text under Eq. (23)).

From (3) and (5) we also have the linearized equations

$$\tau_2 \frac{\partial}{\partial t} \delta P_{xx}^v + \delta P_{xx}^v = - \frac{4}{3} \eta \frac{\partial}{\partial x} \delta v, \quad (9)$$

$$\tau_2 \frac{\partial}{\partial t} \delta P_{ii}^v + \delta P_{ii}^v = \frac{2}{3} \eta \frac{\partial}{\partial x} \delta v \quad (i = y, z), \quad (10)$$

$$\tau_2 \frac{\partial}{\partial t} \delta P_{ij}^v + \delta P_{ij}^v = 0 \quad (i \neq j). \quad (11)$$

Here we will not deal with the general solutions of the system formed by Eqs. (6)–(11). Instead, we look for a special, the simplest possible case such that the study of the evolution of the fluid towards its final equilibrium rest state clarifies the limitations of local-equilibrium thermodynamics. Therefore, by analogy with the case of a rigid system [3], we look for damped oscillatory perturbations. We assume that the perturbations have the simple form

$$\delta v = \delta v_0 \sin(kx) \cos(\omega t) e^{-t/2\tau}, \quad (12)$$

$$\delta \rho = \cos(kx) [A_\rho \sin(\omega t) + B_\rho \cos(\omega t)] e^{-t/2\tau}, \quad (13)$$

$$\delta p = \cos(kx) [A_p \sin(\omega t) + B_p \cos(\omega t)] e^{-t/2\tau}, \quad (14)$$

$$\delta P_{ij}^v = \cos(kx) [A_{p_{ij}} \sin(\omega t) + B_{p_{ij}} \cos(\omega t)] e^{-t/2\tau} \quad (i, j = x, y, z), \quad (15)$$

where $k = 2\pi n/L$, n being a natural number, and $\omega > 0$, $\tau > 0$. We substitute the ansatz (12)–(15) into (6)–(11) and after combining the resulting equations we find

$$A_\rho = -\frac{4k\tau^2\omega}{1 + 4\tau^2\omega^2} \rho_0 \delta v_0, \quad B_\rho = -\frac{1}{2\tau\omega} A_\rho, \quad (16)$$

$$A_p = c^2 A_\rho, \quad B_p = c^2 B_\rho, \quad (17)$$

$$A_{p_{xx}} = -\frac{\tau_2 \omega}{\tau k} \rho_0 \delta v_0,$$

$$B_{p_{xx}} = -\frac{1 - \tau_2/2\tau}{\tau k} \rho_0 \delta v_0, \quad (18)$$

$$A_{p_{ii}} = -\frac{1}{2} A_{p_{xx}}, \quad B_{p_{ii}} = -\frac{1}{2} B_{p_{xx}} \quad (i = y, z), \quad (19)$$

$$A_{p_{ij}} = 0, \quad B_{p_{ij}} = 0 \quad (i \neq j), \quad (20)$$

$$\omega^2 + \frac{1}{4\tau^2} = \frac{c^2 k^2}{1 - \tau_2/\tau}.$$

$$\tau_2 \left(\omega^2 - \frac{3}{4\tau^2} \right) = c^2 k^2 \tau_2 + \frac{4}{3} \frac{\eta k^2}{\rho_0} - \frac{1}{\tau}. \quad (21)$$

Eqs. (21) give ω and τ implicitly in terms of k and the parameters of the fluid c , η , ρ_0 , τ_2 . We note from the first equation in (21) that $\tau > \tau_2$. In the limit $\eta \rightarrow 0$ and $\tau_2 \rightarrow 0$ Eqs. (21) give $\tau \rightarrow \infty$

$\omega \rightarrow ck$, and Eqs. (18)–(20) give $\delta P^v \rightarrow 0$ as they should (see Eq. (3)). Moreover, in the same limit Eqs. (12)–(17) become, as they should, the well-known undamped characteristic vibrations of a fluid in which not only heat conduction but also internal friction is absent [11].

The total energy of the fluid is $E = \int_V \rho(u + v^2/2) dV$, where u is the internal energy per unit mass and V is the total volume of the fluid. The linearized law of balance of total energy gives

$$\frac{dE}{dt} = - \int_V \nabla \cdot [(\rho_0 u_0 + p_0) \delta v] dV = 0. \quad (22)$$

The vanishing of (22) is obtained by writing it as a surface integral and applying the conditions that the velocity is zero at the plane surfaces of the cylinder and that at the curved surface the velocity component normal to this surface vanishes. Eq. (22) shows that our fluid is an isolated system.

Under the local-equilibrium hypothesis one may use the local Gibbs equation which gives

$$s(x, t) - s(x, 0) = -\frac{1}{\rho_0 T_0} \int_0^t dt \delta P_{xx}^v \frac{\partial}{\partial x} \delta v. \quad (23)$$

The left-hand side of (23) can also be written as $\delta s(x, t) - \delta s(x, 0)$. Therefore, from this equation, (12), (15) and (18) we see that δs is of second order in δv_0 , whereas according to (13), (14), (16) and (17) $\delta \rho$ and δp are of first order in δv_0 . This is in accordance with the fact that a term in δs does not appear in Eq. (8). In the limit $\eta \rightarrow 0$ and $\tau_2 \rightarrow 0$ we have $\delta P_{xx}^v \rightarrow 0$ and (23) gives $ds(x, t)/dt = 0$ as it should [11].

Substitution of (12) and (15) into (23), integration in time and using (18) gives, after integration over the whole system,

$$\begin{aligned} \tilde{S}(\tilde{t}) - \tilde{S}(0) &= 1 + \frac{1 - \tilde{\tau}_2}{1 + 4\tilde{\omega}^2} \\ &+ \left(-1 + \frac{1}{2}\tilde{\tau}_2 + \frac{2\tilde{\omega}(1 - \tilde{\tau}_2)}{1 + 4\tilde{\omega}^2} \sin(2\tilde{\omega}\tilde{t}) \right. \\ &\left. + \frac{-2 + \tilde{\tau}_2(1 - 4\tilde{\omega}^2)}{2(1 + 4\tilde{\omega}^2)} \cos(2\tilde{\omega}\tilde{t}) \right) e^{-\tilde{t}}, \quad (24) \end{aligned}$$

where the tildes are used for the set of dimensionless variables

$$\begin{aligned} \tilde{S} &= \frac{4T_0}{\rho_0(\delta v_0)^2 AL} S, & \tilde{\omega} &= \tau\omega, \\ \tilde{\tau}_2 &= \frac{\tau_2}{\tau}, & \tilde{t} &= \frac{t}{\tau} \end{aligned} \quad (25)$$

and we have $\tilde{\omega} > 0$ and $0 < \tilde{\tau}_2 < 1$.

Either from (24) or from (23) we can find the extremal values of $\tilde{S}(\tilde{t})$: for $\tilde{t} = (2n + 1)\pi/2\tilde{\omega}$, with $n = 0, 1, 2, \dots$, which corresponds to vanishing velocity, the entropy reaches a relative maximum; for the values of \tilde{t} such that $\tan(\tilde{\omega}\tilde{t}) = (1 - 2/\tilde{\tau}_2)/2\tilde{\omega}$, which corresponds to vanishing viscous pressure tensor, the entropy reaches a relative minimum.

In order to assess the influence of $\tilde{\omega}$ and $\tilde{\tau}_2$ on the evolution of the reduced entropy \tilde{S} , we plot in Fig. 1 function (24) for a fixed value of $\tilde{\tau}_2$ and several values of $\tilde{\omega}$. Fig. 2 corresponds to a fixed value of $\tilde{\omega}$ and several values of $\tilde{\tau}_2$. In all plots in both figures, the initial entropy is less than the final entropy, which is consistent with the second law of thermodynamics if applied between the initial and the final states. However, in the same figures we observe the non-monotonic evolution of the entropy (24), which has been obtained from the local-equilibrium hypothesis. This is due to the fact that this hypothesis breaks down in situations where linear constitutive laws (such as the Newton law of viscous flow) do not hold [12]. It raises an inconvenience since ac-

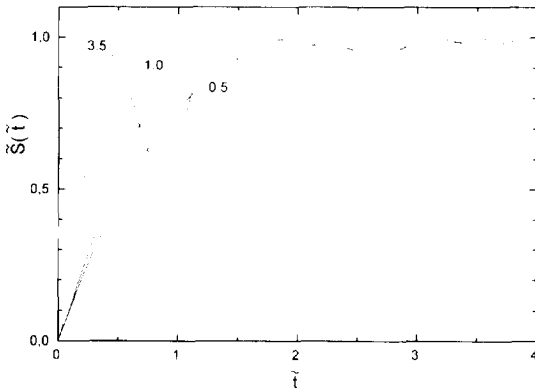


Fig. 1. Reduced entropy as a function of \tilde{t} calculated from local-equilibrium thermodynamics for different values of $\tilde{\omega}$ when $\tilde{\tau}_2 = 0.9$. The origin is fixed at $\tilde{S}(0)$.

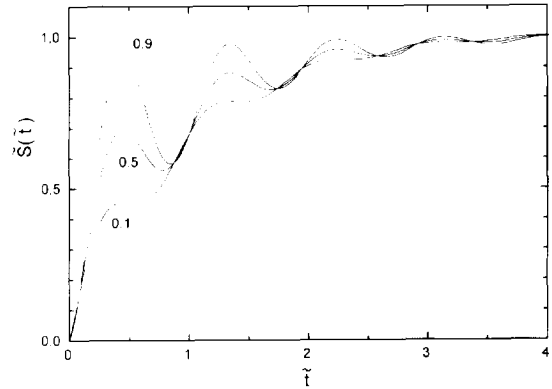


Fig. 2. Reduced entropy as a function of \tilde{t} calculated from local-equilibrium thermodynamics for different values of $\tilde{\tau}_2$ when $\tilde{\omega} = 3.5$. The origin is fixed at $\tilde{S}(0)$.

ording to the second law the entropy of an isolated system can never decrease. We have therefore explicitly seen that for hydrodynamical problems described by equations of the Maxwell–Cattaneo type, which are well founded both theoretically and experimentally, the local-equilibrium formulation of the second law must be generalized. In view of this we will use extended irreversible thermodynamics (EIT), which gives such a generalization.

3. Extended irreversible thermodynamics

EIT does not rest on the local-equilibrium hypothesis. It postulates the specific entropy to depend on the fluxes as well as on the classical variables u and ρ and obtains a specific entropy which reads for a fluid with vanishing bulk viscous pressure in the absence of heat conduction and for small values of the viscous pressure tensor [1]

$$s^*(u, \rho, \mathbf{P}^\nu) = s(u, \rho) - \frac{\tau_2}{4\eta T\rho} \mathbf{P}^\nu : \mathbf{P}^\nu, \quad (26)$$

where $s(u, \rho)$ stands for the local-equilibrium entropy and $\mathbf{P}^\nu : \mathbf{P}^\nu = \sum_{i,j} P_{ij}^\nu P_{ji}^\nu$. In more general situations (26) can be generalized into more complicated expressions, but this will not be necessary to stress the fundamental physical features.

Expression (26) for the entropy may also be obtained from the kinetic theory [5] by using the Boltzmann definition $S = -k \int f \ln f \, dc$, with f the

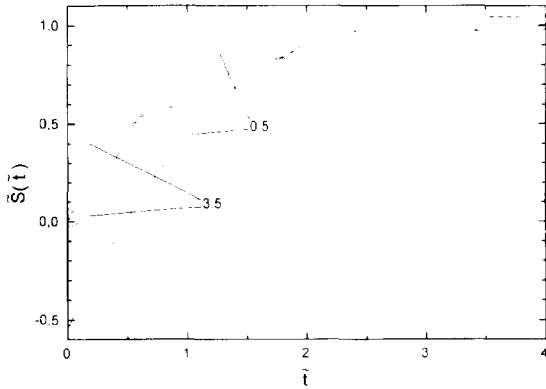


Fig. 3. Reduced entropies as a function of \tilde{t} calculated from local-equilibrium thermodynamics (dashed curves) and from the EIT model (full curves) for different values of $\tilde{\omega}$ when $\tilde{\tau}_2 = 0.9$. The origin is fixed at $\tilde{S}(0)$.

non-equilibrium distribution function and c the molecular velocity. One more way to understand the non-classical term in (26) is by means of the following physical process: consider an elementary cell in a fluid out of equilibrium and take the cell to be sufficiently small so that the velocity gradients can be neglected. If the cell is isolated at some time (which we may take as $t = 0$), it will thereafter decay to equilibrium. The final equilibrium and the initial non-equilibrium specific entropies, s_f and s_i^* , respectively, are related through $s_f - s_i^* = \int dt \sigma / \rho$,

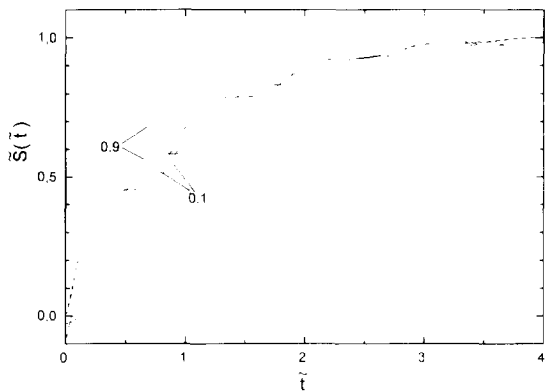


Fig. 4. Reduced entropies as a function of \tilde{t} calculated from local-equilibrium thermodynamics (dashed curves) and from the EIT model (full curves) for different values of $\tilde{\tau}_2$ when $\tilde{\omega} = 3.5$. The origin is fixed at $\tilde{S}(0)$.

where the integration is from $t = 0$ to $t = \infty$ and σ is the rate of entropy production given by

$$\sigma = \frac{1}{2\eta T} h \mathbf{P}^\nu : \mathbf{P}^\nu. \tag{27}$$

Using the exponential decay for \mathbf{P}^ν implied by (3) in this case one readily gets for s_i^* Eq. (26).

In our case (26) can be written, making use of (19) and (20),

$$\rho s^* = \rho s - \frac{3\tau_2}{8\eta T_0} (\delta P_{xx}^\nu)^2. \tag{28}$$

Let us define the difference between both entropies as $s' = s - s^*$. Substitution of (15) and (18) into (28) and integration over the whole system gives, making use of (21) and of (25),

$$\begin{aligned} \tilde{S}'(\tilde{t}) = & \frac{1}{\tilde{\tau}_2 \left(\frac{1}{4} + \tilde{\omega}^2 \right) + 1/\tilde{\tau}_2 - 1} \\ & \times \left[\tilde{\tau}_2 \tilde{\omega} \sin(\tilde{\omega}\tilde{t}) + \left(1 - \frac{1}{2}\tilde{\tau}_2 \right) \right. \\ & \left. \times \cos(\tilde{\omega}\tilde{t}) \right]^2 e^{-\tilde{t}}. \end{aligned} \tag{29}$$

In Figs. 3 and 4 we plot the reduced entropies from EIT and from local-equilibrium thermodynamics for the same values of $\tilde{\omega}$ and $\tilde{\tau}_2$ as in Figs. 1 and 2, respectively. Figs. 3 and 4 show that the total EIT entropy of the isolated system is a monotonic increasing function of time, whereas local-equilibrium thermodynamics cannot be consistently used in situations where the evolution of the fluid is described by relaxational-type equations.

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