Collisions Between Rods: A Visual Analysis

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One of the key subjects in introductory physics is the problem of collisions. It provides a nice example where conservation laws of energy and momentum are essential. Two extreme cases are usually solved: elastic and perfectly inelastic collisions. In the very simple one-dimensional case, velocities before and after collision are readily related through the masses of the colliding bodies. Similar solutions can be found for partially inelastic collisions, provided that the degree of energy loss is known. Otherwise, the energy balance equation cannot be written down. Usually, one can reasonably assess whether the collision is perfectly inelastic (for instance, a bullet impinging onto a piece of wood). However, it is a matter of faith to consider *a priori* a collision as elastic or as being in any intermediate situation. We hope this statement will become clearer to the reader by the end of this paper.

Although everyone knows that some kind of interacting forces are acting during collision, they can be left out of the equation if the kinetic energy after collision is known. However, their nature determines the extent to which energy is conserved. This means that a complete picture of any collision event must take these forces into account. In spite of its interest, such an approach is seldom taken because, in most cases, it is not easy to present at the introductory level. For instance, during collision, macroscopic objects become progressively deformed. In fact, compressive waves are traveling all along the body. On the way, they propagate the initial deformation of the contact surface and ultimately determine the fraction of kinetic energy conserved.[1]

In this paper, we analyze one singular situation where the detailed picture of traveling waves can be described without any kind of mathematical complication. The collision of rods along their axes is so simple that once the way one rod collides against a rigid wall is understood, one has the means to solve graphically a whole series of interesting problems. The solution of selected examples will lead us to the important conclusion that inelastic collisions can occur even in conditions where deformations are completely elastic. Kinetic energy is transferred into elastic waves that make the colliding bodies vibrate while they are separating after collision. Plastic deformation is not necessary at all in order to produce inelastic collisions.

Table I. Velocities after collision against an object at rest.

Elastic collision:		
$v_{1i} = v$	$v_{2i} = 0$	$v_{1f} = (m_1 - m_2)/(m_1 + m_2)v$
$v_{2f} = 2m_1/(m_1 + m_2)v$		
Perfectly inelastic collision:		
$v_{1i} = v$	$v_{2i} = 0$	
$v_{1f} = v_{2f} = m_1/(m_1 + m_2)v$		

Reference Case: Collision Against a Rigid Wall

When an object collides elastically against a rigid wall, we know that as a result it will change the sign of its initial velocity ($m_2 = \infty$, in Table I). This is approximately the case of a small glass ball colliding at moderate velocities with a hard surface. However, many objects do not behave in this same way: Inelastic deformation in rubber or plastic deformation in soft metals can be responsible for energy losses. On the other

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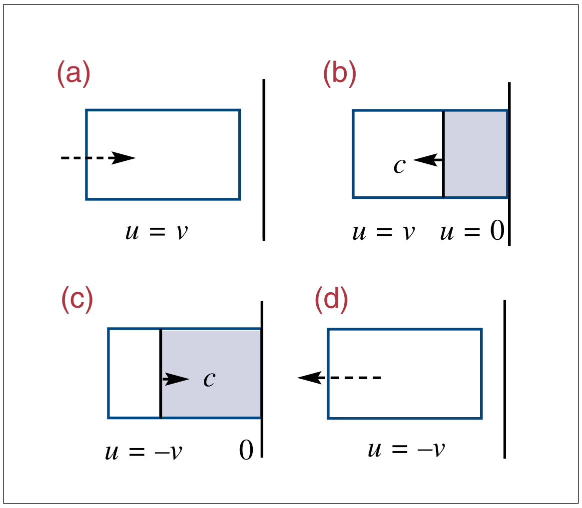

Fig. 1. Collision of a rod against a rigid wall. The uniformly compressed region (shadowed) propagates at the speed of sound, *c*. The local velocity, *u*, changes abruptly from *v* to 0 and –*v* when the deformation front advances.

hand, a rod made of a perfectly elastic material seems a good candidate for an elastic collision. In this section we will give a picture consistent with this hypothesis.

Our starting point is a famous result obtained in the 19th century by B. de Saint-Venant. He demonstrated that the collision of a homogeneous rod of length L against a rigid wall lasts a time, t_c, given by

$$t_c = 2L/c,$$

where c is the speed of sound in the rod.[2] Because elastic waves propagate in solids at speed c, one deduces from Saint-Venant's formula that the collision duration corresponds to the time an elastic wave needs to go forward and back inside the rod. The exact picture, which is otherwise in accordance with the wave equation and boundary conditions,[1] is drawn in Fig. 1. The collision force produces a compressive deformation at the front of the rod [shadowed region in Fig. 1(b)] that propagates at velocity c until the back surface is reached. This occurs at $t_c/2$ when the whole rod is at rest and homogeneously compressed. Kinetic energy has been converted into elastic potential energy. Now, elastic energy begins to relax at the free surface of the rod [Fig. 1(c)]. This relaxation also can be understood in terms of the propagation of elastic waves. Reflection of the compressive initial wave results in a tensile wave that exactly compensates for the previous compression and allows the rod to progressively recover its initial speed. The collision finishes when the front of the reflected wave reaches the wall. At this moment, no elastic energy remains and the local velocity, u, at any point in the rod is just $-v$. The rod leaves the wall without loss of kinetic energy [Fig. 1(d)]. So, this picture explains that the collision of a rod against a rigid wall is elastic. Although readers may think that they "intuitively" knew it was, perhaps they might begin to have doubts about their intuition if they knew that, in the case of the section not being uniform, the collision would no longer be elastic.

Collision of Two Identical Rods

The problem now is to deduce what the final speeds will be when one rod collides against an identical one at rest. We know that, if the collision is elastic, the rods will simply exchange their velocities (Table I). The following analysis, based on the picture developed in the previous section, will demonstrate that this is the case.

Let us consider the problem from a frame attached to the center of mass (c.m.). Due to the fact that the two rods are identical, their center of mass will always be midway between them. This means that, during collision, the c.m. will behave like a rigid wall for each rod. So, this problem is equivalent to the previous one and the collision is indeed elastic. This event is schematically shown in Fig. 2. The local speed relative to the c.m., u_{cm}, is deduced from Fig. 1 with the slight difference that now each rod approaches the equivalent rigid wall (their c.m.) at $v/2$ and not v. For our purposes, it is interesting to pay attention to the values this speed takes in the lab frame, u. During the collision time the contact surfaces move together at half the initial speed, $v/2$, and kinetic energy is progressively transferred from one rod to the other by the deformation fronts moving at speed c [Fig. 2(b)]. In fact, one clearly sees that kinetic energy is conserved because both fronts reach the contact surface simultaneously [Fig. 2(c)] and any deformation disappears [no shadowed region remains in the rods after the collision, Fig. 2(d)].

Now, consider a collision between two rods of

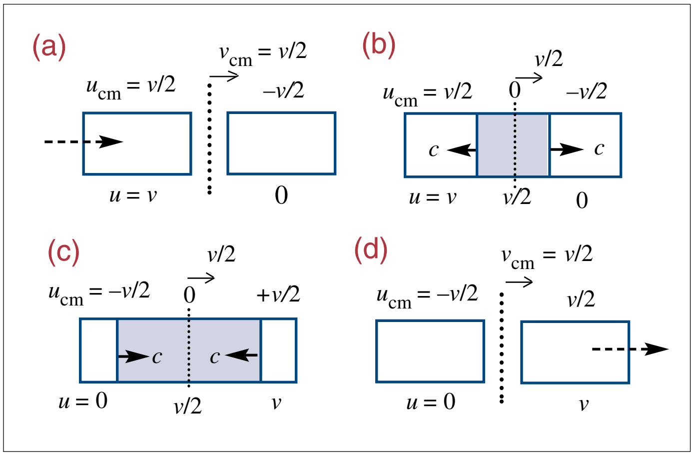

Fig. 2. Collision of two identical rods. The plane of the center of masses (c.m.) is equivalent to a rigid wall. In this frame the problem is reduced to that of Fig. 1. The lack of any compressed (shadowed) region in (d) means that collision has been elastic.

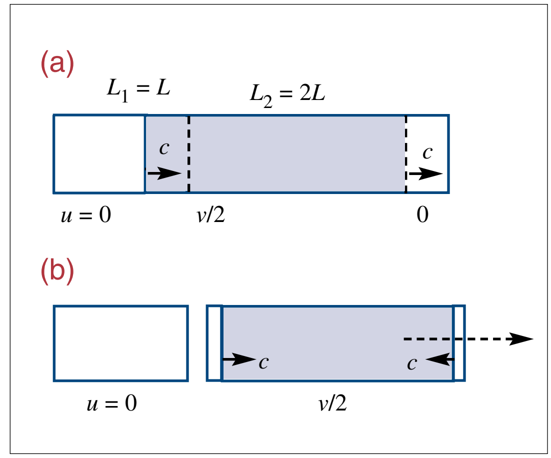

Fig. 3. Last steps of the collision against a rod of double length. After collision (b), the longest rod is uniformly compressed. Only one-half of the initial kinetic energy is conserved.

equal density and dimensions but made of materials with different c values (for instance, steel and some kind of copper alloy). It will not be elastic because the deformation fronts will not reach the contact surface at the same time, and a fraction of the initial kinetic energy will be stored as deformation.

Collision of Rods with Different Lengths

This case constitutes a straightforward extension of the previous example. Because of the different lengths, the deformation fronts will not return simultaneously to the contact surface. So, the collision will be inelastic. Let us have a look at the evolution of the elastic deformation, as shown in Fig. 3, for the particular case in which the rod initially at rest is twice as long as the moving one of length L.

Initially, the picture of propagating compressive deformation will be exactly the same as in Fig. 2(b). This is so because when the rods begin to collide, the interaction at the contact surface is exactly the same, irrespective of rod length. The elastic compression depends only on local parameters such as density, elastic modulus, and initial velocity.[1,2] After time $t = L/c$, the wave front in the impinging rod is reflected, whereas in the longest rod it has still not reached the end [Fig. 3(a)]. So, the impinging rod is progressively losing its elastic and kinetic energy, whereas the other rod is brought progressively into movement and compressed. After a time $t = 2L/2$ (that is, just after collision), the impinging rod is completely at rest whereas the longest one is moving at half the initial speed [Fig. 3(b)]. This value ensures momentum conservation. However, the final kinetic energy is only half the initial value. The question is now, where has the remaining half gone? It is simply stored, homogeneously, as an elastic deformation in the longer rod, which is compressed [shadowed region in Fig. 3(b)]. Once the collision has finished, this elastic energy makes the rod vibrate, due to continuous propagation and reflection of the deformation fronts.[3] A similar analysis would deliver the kinetic energy lost when the ratio of rod lengths has an arbitrary value.[4]

Collision Against a "Rod-Chain"

The last example will be one rod impinging against an arrangement of two identical rods in contact, at rest, as in Fig. 4. The trivial solution corresponds to the situation where the third rod leaves the ensemble at the initial speed. However, even if we knew that energy was conserved, the equations of energy and momentum conservation are not able to determine uniquely this solution because there are more unknowns (the final speeds of three bodies) than equa-

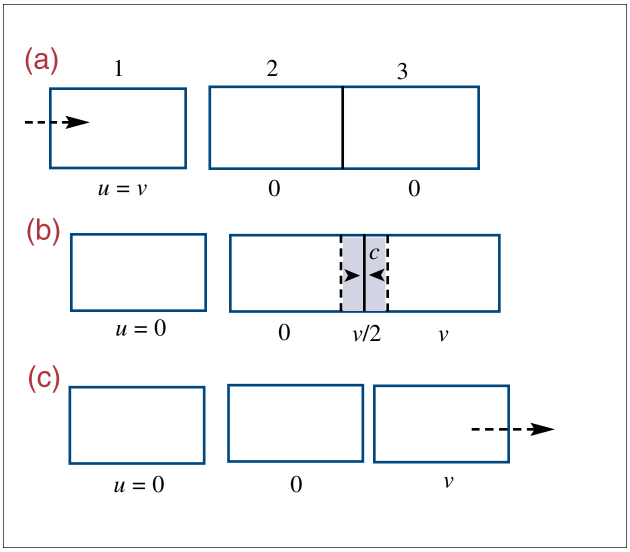

Fig. 4. Collision against two identical rods in contact. The elastic energy accumulated in the two initially static rods is finally transferred as kinetic energy to the last one.

tions (two). In the following, we will show that the graphic method developed so far is able to confirm that this is in fact the correct solution.

The evolution of the deformation is identical to the previous example up to the point where the impinging rod is left at rest [Fig. 3(b)]. Afterwards, rods 2 and 3 move away together at half the initial speed, and relaxation of the compressive stress begins at the opposite surface of the rods [compare Fig. 3(b) to Fig. 2(b)]. When the deformation fronts meet at the contact surfaces [Fig. 4(b)], the central rod remains at rest and the last one moves with the initial speed. No deformation is left in any rod (no shadowed region remains). So, the collision has been elastic [Fig. 4(c)]. The result would be exactly the same if an arbitrary number of identical rods were placed in the chain. Furthermore, one intermediate rod of arbitrary length would also transmit the energy between the impinging rod and the final one without losses.

This example constitutes a very simple version of the ball-chain (also known as Newton's Cradle) experiment. If rods were substituted with identical balls, the result would be almost the same.[5] A similar behavior is observed with other objects such as coins or even nuts (!). It is always the last piece in the chain that takes the greatest part of kinetic energy. The calculation of interaction forces is, in all such cases, very complicated except in the rod-chain. Thus, our example provides a very simple demonstration of a more general phenomenon.

Conclusion

We have shown that a complete understanding of the collision between extended objects must take into account the deformations produced during collision. In general, these deformations are propagated as elastic waves inside the objects and are responsible for the energy exchange. If deformations are elastic, the collision will only be elastic if the elastic waves cancel out once the collision is finished. This condition is very difficult to achieve. So, we can conclude with a somewhat contradictory sentence: "Collisions are in general nonelastic, even in the elastic regime (of deformations)."

Acknowledgments

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References

1. See, for instance, D. Auerbach, "Colliding rods: Dynamics and relevance to colliding balls," *Am. J. Phys.* **62**, 522–525 (June 1994).
2. A sound derivation of this formula can be found in P. Roura, "Collision duration in the elastic regime," *Phys. Teach.* **35**, 435–436 (Oct. 1997).
3. A nice consequence of this analysis is that the colliding rods will ring only in cases where they have different lengths (i.e., partially inelastic collisions); see F.M.F. Mascarenhas, C.M. Spillmann, J.F. Lindner, and D.T. Jacobs, "Hearing the shape of a rod by the sound of its collision," *Am. J. Phys.* **66**, 692–697 (Aug. 1998).
4. It is left for the reader to derive that the longer the second rod, the greater the loss of kinetic energy. For $L_2 > L_1$, the fraction of the initial kinetic energy lost is $1 - L_1 / L_2$.
5. A number of papers devoted to the ball-chain problem are reviewed in J.D. Gavenda and J.R. Edgington, "Newton's Cradle and scientific explanation," *Phys. Teach.* **35**, 411–417 (Oct. 1997).

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