Energy conservation can predict intervals of time

Consider a simple energy problem. A mass m, which is initially at rest, falls down due to Earth's gravity ($g = 9.8$ m/s^2). Friction is negligible. Conservation of energy states:

$$K_0 + P_0 = K + P \tag{1}$$

where K_0 and P_0 are the kinetic and potential energies, respectively, at the initial position (height h), whereas K and P are the corresponding energies when the mass reaches the ground (height zero). Equation (1) can also be written as

$$mgh = \frac{mv^2}{2}, \tag{2}$$

which yields the well-known result for the final speed $v = \sqrt{2gh}$.

Now consider an observer moving with constant speed V relative to the ground. At first sight, we would argue that conservation of energy (like all physical laws) must hold true in any frame of reference, and that for this observer equation (1) can be written as

$$\frac{m(-V)^2}{2} + mgh = \frac{m(v-V)^2}{2} \tag{3}$$

where $v - V$ is the final speed. In the first part of equation (3) we have taken into account that the initial speed is $-V$ in this frame.

From equation (3) we expect to obtain $v = \sqrt{2gh}$, i.e. the same result as from equation (2). However, it yields a different result! How can this be solved? The solution follows. I also found that it has an application to a problem that is usually considered impossible to solve by applying the conservation of energy!

Solving the problem

In the frame where the observer is moving downwards with a constant speed V, the final height of mass m is not zero, but Vt (because the observer has moved a distance Vt down during time t of the fall). So, equation (3) is not the right form of equation (1). We have to add an additional term,

$$\frac{m(-V)^2}{2} + mgh = \frac{m(v-V)^2}{2} + mgVt\,. \tag{4}$$

This equation can be rewritten as

$$mgh - mgVt = \frac{mv^2}{2} - mvV. \qquad (5)$$

Relativity requires that the laws of physics hold true in all frames. This equation must hold for any value of V. Obviously this is possible only if the terms independent of V are the same, and the terms proportional to V are also equal. Thus

$$mgh = \frac{mv^2}{2} \qquad (5)$$

$$-mgVt = -mvV. \qquad (6)$$

Using equation (5) yields the right result: $v = \sqrt{2gh}$. The problem is solved. Whichever frame we consider, we obtain the same answer for the final speed v relative to the ground (despite the fact that the kinetic energy depends on the frame).

But we can go further. We can use equation (6) to predict the time it takes the mass m to hit the ground. Indeed, we obtain $t = v/g = \sqrt{2h/g}$. Many textbooks stress that energy conservation is useful to determine speed and distance, but not time [1]. Resolution of a simple problem has made it possible to realize that, in fact, energy conservation can also be used to predict time intervals.

Reference

[1] Tipler P A 1991 *Physics for Scientists and Engineers* vol. I (New York: Worth) p156

J Fort *Departament de Física, Edifici P-II, Universitat de Girona, 17071 Girona, Catalonia, Spain, e-mail: joaquim.fort@udg.es*