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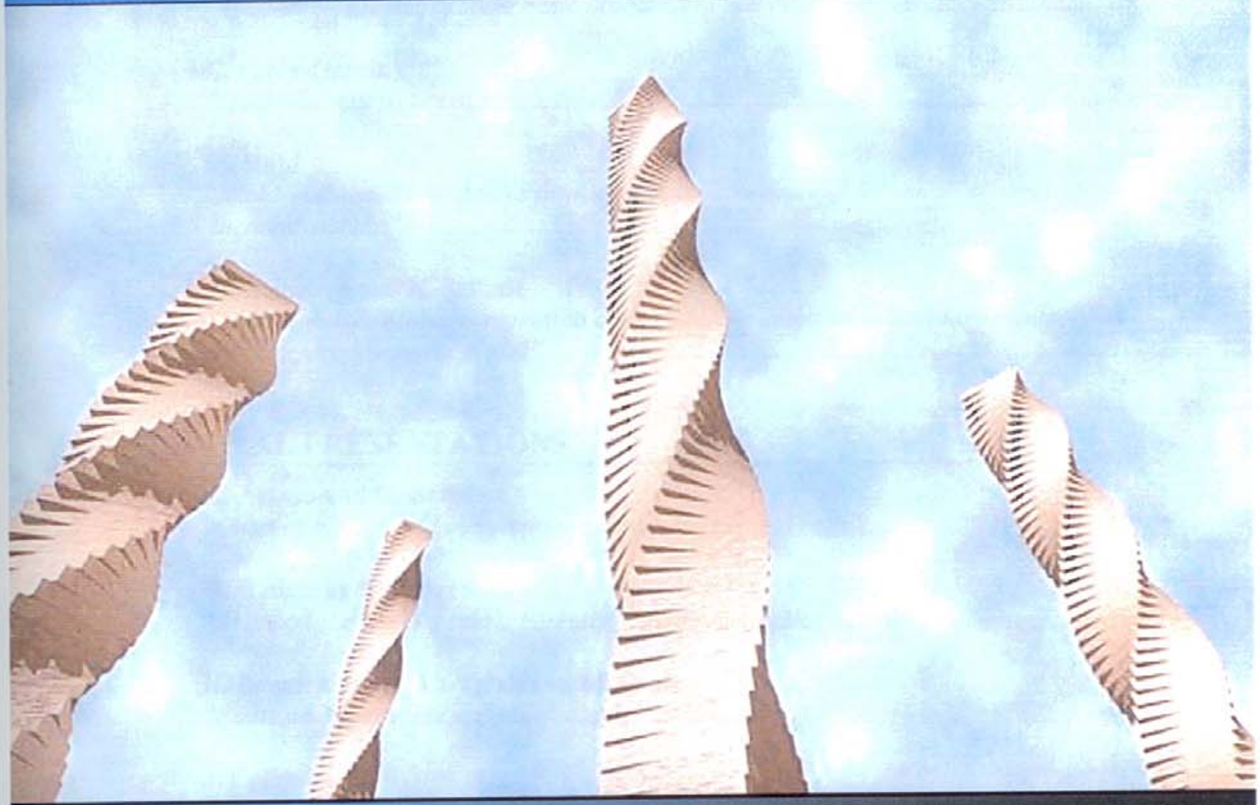

**Universitat Autònoma
de
Barcelona**

**Centre Européen de Réflexion
et
d'Etude en Thermodynamique**

PHYSICAL MODELS OF BIOLOGICAL SYSTEMS. THEORY VERSUS EXPERIMENTS

Joaquim Fort[1], Vicenç Méndez[2], Daniel Campos[3] and Vicente Ortega-Cejas[3]

[1] Dept. de Física, Univ. de Girona, Campus de Montilivi, 17071 Girona, Spain

[2] Dept. de Medicina, Univ. Internacional de Catalunya, 08190 Sant Cugat del Vallés, Spain

[3] Dept. de Física, Univ. Autònoma de Barcelona, E-08193 Bellaterra, Barcelona, Spain

A. THERMODYNAMIC DERIVATION

Before discussing some specific applications [1], let us derive the most basic necessary equations. The second law of thermodynamics has strong implications on the possible equations driving the dynamics of physical and biological systems. In Extended Irreversible Thermodynamics [2], one assumes that the entropy density s depends on the classical variables and also on the dissipative fluxes. Therefore, in diffusive systems

$$s = s(n, \vec{J}), \qquad (1)$$

where n is the number density of particles and $\vec{J}$ the particle number flux. The entropy differential of any process is thus

$$ds = -\frac{\mu}{T}dn - \frac{\alpha}{Tn}\vec{J}\cdot d\vec{J}, \qquad (2)$$

where T is the temperature, μ is the chemical potential per particle and α is a scalar (which may depend on n and T, but is independent of $\vec{J}$ at this order of approximation). On the other hand, the particle balance equation is obviously

$$\frac{\partial n}{\partial t} = -\vec{\nabla}\cdot\vec{J} + F(n), \qquad (3)$$

where $F(n)$ is the number of particles (or individuals, in biological applications) generated per unit time and volume (or per unit area, in 2-dimensional systems).

From Eqs. (2) and (3), and comparing with the balance law for the entropy, namely

$$\dot{s} + \vec{\nabla}\cdot\vec{J}^s = \sigma_{syst}, \qquad (4)$$

we find for the entropy production rate per unit volume of the system (with $\vec{J}^s = -\frac{\mu}{T}\vec{J}$ as usual [2])

$$\sigma_{syst} = -\frac{\vec{J}}{T}\cdot\left[\vec{\nabla}\mu + \alpha\frac{\partial \vec{J}}{\partial t}\right] - \frac{\mu F(n)}{T}, \tag{5}$$

where we have assumed that the temperature is uniform for simplicity. Now we note that the physical volume contains not only the particles (or individuals), which is all we have so far considered, but also the medium (or environment) which is necessary to generate new particles. This need follows simply from the principle of mass conservation. According to the second law, what must be semipositive-definite is the total entropy production, namely $\sigma_T = \sigma_{syst} + \sigma_{env}$, i.e. the sum of the entropy production of both subsystems (and not each of them separately). Thus we have

$$\sigma_T = -\frac{\vec{J}}{T}\cdot\left[\vec{\nabla}\mu + \alpha\frac{\partial \vec{J}}{\partial t}\right] - \frac{\mu F(n)}{T} + \sigma_{env} \geq 0 \quad . \tag{6}$$

There are two processes: particle diffusion and particle generation (the latter is called reproduction in biology). Each physical *process* must have a semipositive-definite entropy production rate,

$$\sigma_{diff} = -\frac{\vec{J}}{T}\cdot\left[\vec{\nabla}\mu + \alpha\frac{\partial \vec{J}}{\partial t}\right] \geq 0, \tag{7}$$

$$\sigma_{env} - \frac{\mu F(n)}{T} \geq 0. \tag{8}$$

The simplest way to ensure Eq. (7) is that the bracket is equal to $L\,\vec{J}$, with L a proportionality constant. This yields the Maxwell-Cattaneo equation

$$\tau\frac{\partial \vec{J}}{\partial t} + \vec{J} = -D\vec{\nabla}n, \tag{9}$$

where $\tau \equiv \alpha / L$ is called the relaxation time and $D \equiv (\partial\mu / \partial n)/L$ the diffusion coefficient. Combining Eqs. (9) and (3), we obtain the so-called hyperbolic reaction-diffusion equation

$$\tau\frac{\partial^2 n}{\partial t^2} + \frac{\partial n}{\partial t} = D\,\vec{\nabla}^2 n + F(n) + \tau\frac{\partial F(n)}{\partial t}. \tag{10}$$

Thermodynamics is a very powerful way to derive the possible forms of the transport equations, and can be applied to any system, either physical or biological, and whatever the microscopic mechanism of motion at work.

B. COMPARISON TO OBSERVATIONS

In order to compare to observations, we need a prediction for the diffusion parameters D and τ, in terms of directly measurable quantities. A random-walk approach yields the usual result $D = <\Delta^2>/(4T)$ (in two dimensions), where $<\Delta^2>$ is the mean square displacement of the particles per jump and T the mean time interval between successive jumps, and [3]

$$\tau = \frac{T}{2}. \qquad (11)$$

Eqs. (10)-(11) have front solutions, which are nothing but invasion waves, i.e. profiles of population density $n(r-ct)$ with a constant S-shape. In one side of the front, towards which it moves, the system is in the state $n=0$ (corresponding to an empty environment). In the other side, which has been already swept by the front, the system is in the final state $n = n_{\max}$ (corresponding to saturated population density). Fronts satisfying Eqs. (10)-(11) move at the constant speed [1]

$$c = \frac{2\sqrt{aD}}{1+\frac{aT}{2}}, \qquad (12)$$

where $a \equiv \left.\frac{dF}{dn}\right|_{n=0}$.

B.1. ANTHROPOLOGY

The first comparison of Eq. (12) to observations was made for the Neolithic transition in Europe [3]. Archaeological data (circles in Fig. 1) show that agriculture arose in the Near East about 9,000 years ago, from it gradually spread over Europe at a speed of 1.0 ± 0.2 km/yr. Eq. (12) predicts a speed consistent with this range, whereas previous models predicted a higher speed (Fig. 1). The delay time T appearing in Eq. (12) is the mean generation time in this case.

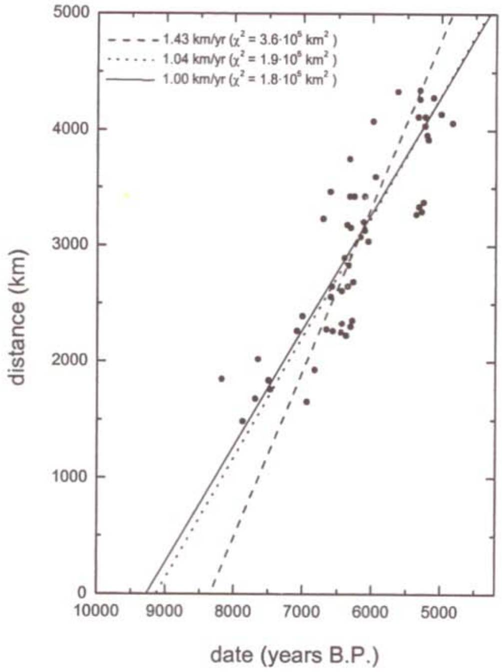

Fig. 1 The Neolithic transition in Europe [3]. Data were first collected by Ammerman and Cavalli-Sforza. Distances are measured from Jericho (the presumed centre of diffusion). Dates are in years before present. The full line is a linear regression fit to the data points. The dotted line is a least-square fit with slope computed from Eq. (12), and the dashed one from Fisher's model [Eq. (12) with $\tau = 0$].

B.2. BIOLOGICAL INVASIONS

Eq. (12) has been also compared to data from other population invasions [4], yielding better agreement than the parabolic or Fisher approximation (Fig. 2).

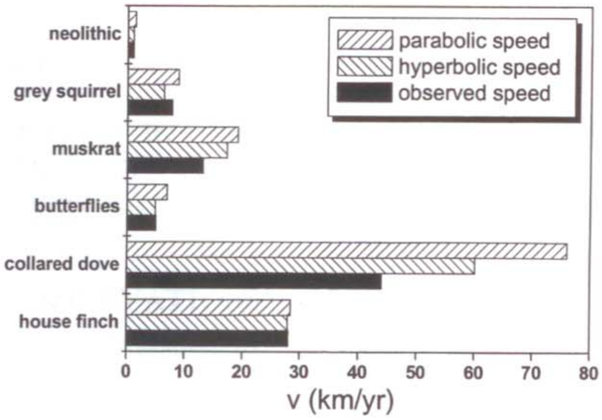

Fig. 2 Theoretical versus observed speeds for several biological invasions [4].

B.3. SPREAD OF VIRUS INFECTIONS

Eq. (10) can be generalized to cases in which several species coexist. One such example, for which it has been possible to understand previously-unexplained experimental data, is the cases of virus infections [5]. Fig. 3 presents an example of the comparison between theory and experiment. No free or adjustable parameters at all have been used, whereas previous approaches used three adjustable parameters and equations which did not take into account the effect of the relaxation time τ (i.e., they considered $\tau = 0$). In this case, this effect is due to the time it takes for a virus to reproduce and kill a host cell, after entering into it. Once a cell is killed, the progeny of the virus leave it and diffuse until they reach new cells. Then, the cycle begins again. Thus, the front spreads.

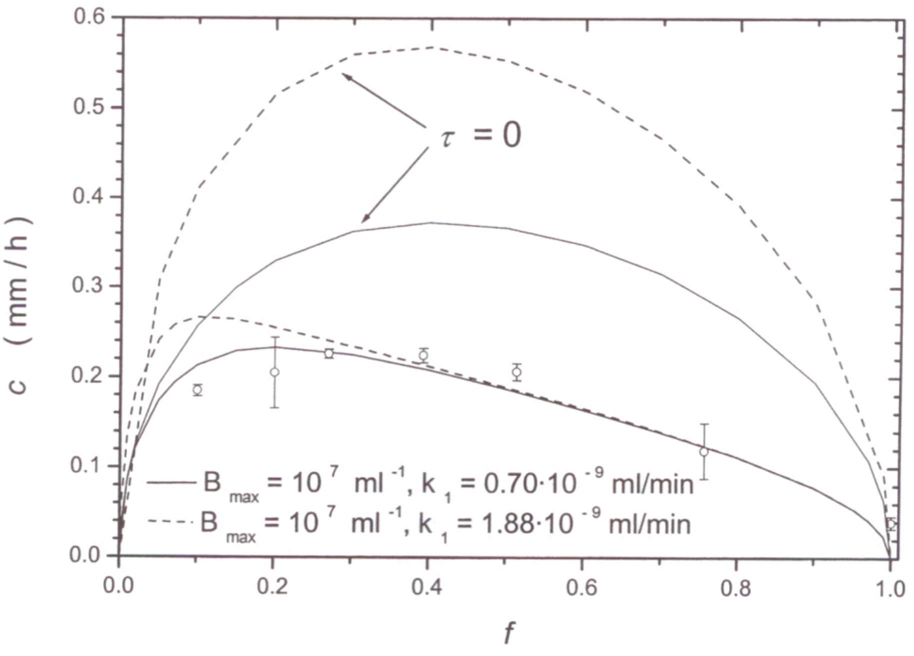

Fig. 3 *Theory (lines) versus experiment (open circles with error bars) for the speed of virus infections [5]. Predictions are shown for two extreme values of a kinetic rate parameter* k_1 *[which has a role similar to F(n) in Eq. (3)], as determined from independent experiments. f is the asymptotic concentration of cells relative to its maximum possible value* B_{max}*, which in turn depends on the nutrient concentration used in the experiment.*

B.4. ECONOMICS

Several other extensions of the basic Eq. (10) are possible. A discrete approach has been used to explain the income distribution function from the intergenerational income transmission probabilities in several countries [6].

B.5. UNEXPLAINED EXPERIMENTAL OBSERVATIONS

The speed of forest colonizations and of some nervous impulses are examples of experimental observations of fronts with, still today, await theoretical explanation.

REFERENCES

[1] J. Fort and V. Méndez, Repr. Progr. Phys. **65**, 895 (2002).

[2] D. Jou, J. Casas-Vázquez and G. Lebon, *Extended Irreversible Thermodynamics* 3rd ed. (Springer, Berlin, 2001).

[3] J. Fort and V. Méndez, Phys. Rev. Lett. **82**, 867 (1999). http://www.aip.org/physnews/graphics/html/diffuse.htm

[4] D. Campos, J. Fort and J.E. Llebot, Phys. Rev. E **66,** 062901 (2002); V. Ortega-Cejas, J. Fort and V. Méndez, Ecology, in press (2003).

[5] J. Fort and V. Méndez, Phys. Rev. Lett. **89**, 178101 (2002). http://www.aps.org/media/tips/tips102302.html

[6] J. Fort and V. Méndez, submitted (2003).